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# AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

## PAPERS

### THE MECHANISM OF ENERGY LOSS IN FLUID FRICTION

BY BORIS A. BAKHMETEFF,<sup>1</sup> M. AM. SOC. C. E., AND  
WILLIAM ALLAN,<sup>2</sup> ASSOC. M. AM. SOC. C. E.

#### SYNOPSIS

An account of the inner processes, by which the energy of fluid flow is lost through friction, is presented in this paper. Previous investigations have dealt largely with bulk appraisals, relating to over-all losses of head. The internal mechanism was investigated in pioneer fashion by Sir G. G. Stokes,<sup>3</sup> who first expressed the internal energy balance for motion in viscous media and formulated the so-called "dissipation function." Horace Lamb<sup>4</sup> broadened the Stokes reasoning, applying it to laminar motion in general. More recently, an attempt to interpret energy exchange in turbulent motion was made by G. I. Taylor.<sup>5,6</sup>

Although it is unimpeachable from a formal mathematical point of view, the traditional treatment does not elucidate the physical aspect of the problem, failing in particular to reveal the fundamental fact that the withdrawal of energy from the flow and its final dissipation into molecular heat do not coincide spatially and that they constitute the initial and the ultimate phase of a manifold process.

The paper reveals these consecutive phases, explaining their dynamic significance and the nature of the losses involved in the different stages. The facts disclosed are especially pertinent for gaining a more comprehensive understanding of the mechanism of turbulent flow. The analysis begins with friction in uniform established motion and subsequently is extended to boundary layer

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by July 1, 1945.

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<sup>3</sup> "On the Effect of Internal Friction of Fluids on the Motion of Pendulums," by Sir G. G. Stokes, *Transactions, Cambridge Philosophical Soc.*, Vol. IX, 1850, p. 8; also the collected papers in Vol. III, p. 7.

<sup>4</sup> "Hydrodynamics," by Horace Lamb, 6th Ed., Univ. Press, Cambridge, England, 1932, p. 579.

<sup>5</sup> "Statistical Theory of Turbulence," by G. I. Taylor, *Proceedings, Royal Soc. of London, Series A*, Vol. CLI, 1935, pp. 455-464.

<sup>6</sup> "Modern Developments in Fluid Dynamics," edited by S. Goldstein, Clarendon Press, Oxford, England, 1938, Vol. II, p. 395.

flow. The fundamental relations are established first for the simpler case of laminar motion, and subsequently are applied to the more complex turbulent flow.

### NOTATION

The letter symbols in this paper, defined in the text where they first appear, conform essentially to American Standard Letter Symbols for Hydraulics (ASA—Z10.2—1942), prepared by a Committee of the American Standards Association, with Society representation, and approved by the Association in 1942.

### 1. INTRODUCTION TO UNIFORM RECTILINEAR MOTION

Fluid friction is revealed outwardly by the resistance slope  $S = h_f/l$ , or the pressure gradient

$$-\frac{dp}{dx} = \frac{p_1 - p_2}{l_{1,2}} = \gamma \frac{h_1 - h_2}{l_{1,2}} = \gamma S \left( \frac{\text{lb} \times \text{ft}}{\text{ft}^3 \text{ per ft}} \right) \dots \dots \dots (1)$$

in which:  $p$  = pressure intensity;  $l$  = distance measured in the direction of flow;  $\gamma$  = specific weight (weight per unit volume); and  $h$  = head, measured from a datum line. In this instance Eq. 1 measures the rate of energy per unit

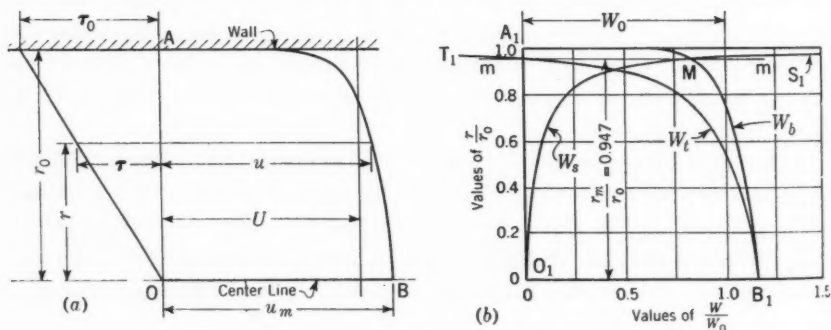


FIG. 1.—TURBULENT FLOW IN SMOOTH PIPE

volume lost equally by each and every element of a cross section, in its displacement over a unit length of the conduit. The same quantity (Eq. 1) serves to determine the mean wall friction stress

$$\bar{\tau}_o = -\gamma S R = \frac{dp}{dx} R \left( \frac{\text{lb}}{\text{ft}^2} \right) \dots \dots \dots (2)$$

and, with that, the internal shearing stress structure. The latter, in uniform motion, is linear.<sup>7</sup> Thus, for a pipe (Fig. 1)—

$$\tau = \tau_o \frac{r}{r_o} \dots \dots \dots (3a)$$

<sup>7</sup> "The Mechanics of Turbulent Flow," by B. A. Bakhmeteff, Princeton Univ. Press, Princeton, N. J., 1936, p. 3.

and, for two-dimensional motion (Fig. 2)—

$$\tau = \tau_o \frac{y'}{y'_o} \dots \dots \dots (3b)$$

In Eqs. 2 and 3,  $r$  = the distance from the axis to an inner filament;  
 $A$  = the cross-sectional area;  $P_w$  = the wetted perimeter;  $R = A/P_w$  the

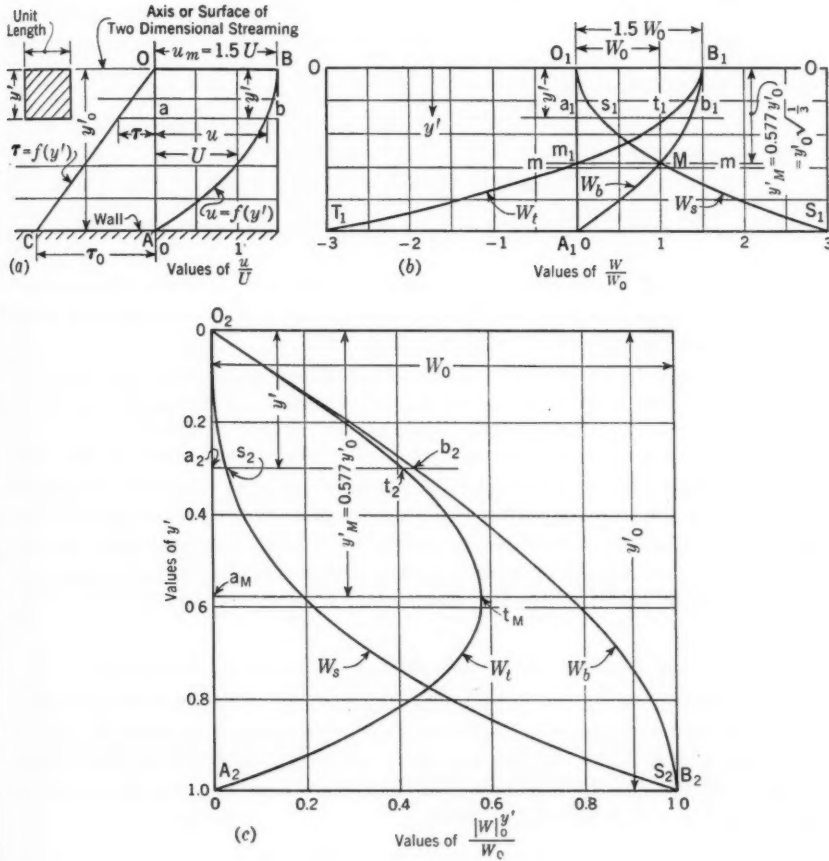


FIG. 2

hydraulic radius; and  $y'$ , in two-dimensional motion, will designate specifically the distance from the free surface (Fig. 2) or from the central axis as against  $y$ , which exemplifies a transverse coordinate in general.

Another familiar concept is the bulk loss of power:

$$W_p = \gamma S Q = - \frac{dp}{dx} Q \left( \frac{\text{lb} \times \text{ft}}{\text{sec per ft}} \right) \dots \dots \dots (4)$$

or the total work done by the flow in overcoming frictional resistances during unit time and over a unit length of the conduit. In Fig. 3, the discharge  $Q$

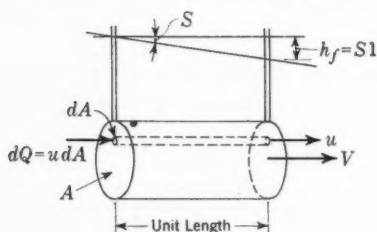


FIG. 3

enters the unit length block and subsequently leaves it, in the interim unit time suffering the loss  $W_o$ ; or, one may state that within the unit length block, and during each second, mechanical energy at the rate of  $W_o$  is withdrawn from the flow and is spent to overcome resistances, by way of ultimate dissipation into molecular heat. Since the velocities do not change, all of  $W_o$  is

withdrawn from the potential energy store carried by the flow, thereby causing a reduction of the piezometric head by  $h_f = S \times 1$ . The foregoing may be symbolized by writing

$$W_o = (W_b)_o = (W_s)_o \dots \dots \dots (5)$$

with the subscript  $b$  designating the process of withdrawal or the "borrowing" of the energy from the flow, and the subscript  $s$  indicating the spending of the energy on resistances.

The equality shown in Eq. 5 holds for bulk relations only. In fact, contrary to Eq. 5, the local values of  $w_b$  and  $w_s$  are generally not equal and do not balance each other. Indeed, the cardinal fact which dominates the entire energy loss mechanism is that the cross-sectional distribution of the local borrowing quantity  $w_b$  and the local spending rate  $w_s$  manifest altogether different, and in a sense opposite, outlines, and that for such reason the borrowing and the spending phases are necessarily linked by an intermediate process, the function of which is to transmit energy withdrawn at one part of the cross section to other parts, where the mechanism of energy exchange calls for its spending.

## 2. THE ENERGY WITHDRAWING OR BORROWING FUNCTION

The distribution of the local borrowing rate  $w_b$  is determined by noting that the energy withdrawn per unit time within an elementary tube  $dA$  (Fig. 3) is  $\gamma S dQ = \gamma S u dA$ , with  $u$  the local velocity. Dividing by the volume  $dA \times 1$ , one obtains the unit rate (that is, the work over a unit time interval, apportioned to a unit volume of fluid):

$$w_b = \gamma S u = - \frac{dp}{dx} u \left( \frac{\text{lb} \times \text{ft}}{\text{ft}^3 \times \text{sec}} \right) \dots \dots \dots (6)$$

which shows that the spatial distribution of  $w_b$  follows the local velocity profile, copying the latter to the scale  $\gamma S$  (curves BA and B<sub>1</sub>A<sub>1</sub> in Figs. 2(a) and 2(b)).

In two-dimensional flow, for a block of unit width normal to the plane (Fig. 2), the total borrowed energy will be

$$(W_b)_o = \gamma S \int_0^{y'_o} u dy' = \gamma S U y'_o = \gamma S q_o = W_o \dots \dots \dots (7)$$

and the average cross-sectional unit rate will be

$$w_o = \frac{W_o}{y'_o} = \gamma S U = - \frac{dp}{dx} U \dots \dots \dots (8)$$

Furthermore, the aggregate energy amount "withdrawn" during a unit time from a partial cross-sectional block, such as that shown in Fig. 2(a) by the hatched area, is

$$|W_b|_0^{y'} = \int_0^{y'} w_b dy' = \gamma S \int_0^{y'} u dy' = \gamma S |q|_0^{y'} \dots \dots \dots (9)$$

constituting the "cumulative" borrowing rate between the transverse limits, as indicated. The quantity  $|q|_0^{y'}$ , measured by the area OBba in Fig. 2(a), will be termed the "cumulative discharge." The corresponding quantities  $|W_b|_0^{y'}$  are defined in Fig. 2(c) by curve  $O_2b_2B_2$ , with the coordinate  $a_2b_2$  representing the area  $O_1B_1b_1a_1$  under the part  $B_1b_1$  of the local  $w_b$ -curve in Fig. 2(b). Obviously, for  $y' = y'_o$ ,  $|W_b|_0^{y'}$  equals  $W_o$  as defined in Eq. 7.

**Differential Form.**—To express the energy borrowing process mathematically, note that for a two-dimensional element (Fig. 4), the

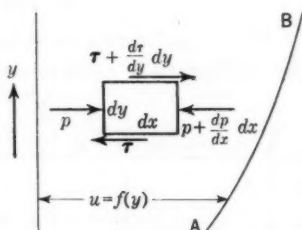


FIG. 4

resulting friction force is  $\left( \tau + \frac{d\tau}{dy} dy \right) dx - \tau dx = \frac{d\tau}{dy} dx dy$ . In uniform established motion frictional traction is in balance with the pressure resultant  $\left( - \frac{dp}{dx} dx \right) dy$  which, apportioned to a volume unit, defines the equilibrium premises:

$$- \frac{dp}{dx} + \frac{d\tau}{dy} = 0 \dots \dots \dots (10a)$$

and

$$\frac{d\tau}{dy} + \gamma S = 0 \dots \dots \dots (10b)$$

The sign of the stress gradient  $d\tau/dy$  is necessarily negative, for the friction

resultant opposes the motion and compensates the positive pressure, or the gravity force, actuating the flow. The stress  $\tau$  is essentially an internal agency. Action on the interfaces n and m of adjoining filaments (Fig. 5(b)) is equal in size, but opposite in direction. In determining which sign (plus or minus) is to be used for  $\tau$  in Eqs. 10 and subsequently, one is guided by the direction of

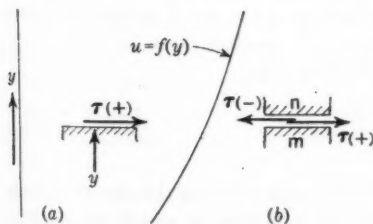


FIG. 5

the traction on the face of a fluid block (Fig. 5(a)), terminated by the positive ordinate  $y$ . As is evident, the sign of  $\tau$  will coincide appropriately with that of  $du/dy$ .

The unit time work performed by friction on the element (Fig. 4) is  $\frac{d\tau}{dy} dx dy u$ , which is also a negative quantity to be compensated by the positive work of the pressure resultant  $[(-dp/dx) dx dy u]$ . The latter, in turn, is equal to the positive unit rate at which energy is withdrawn locally from the flow. Hence

$$w_b = -\frac{dp}{dx} u = -u \frac{d\tau}{dy} \dots \dots \dots (11)$$

Note that, thus far, no limitations have been imposed on the particular nature of the stress-generating agencies. Therefore, Eq. 11 is quite general and will apply to all cases of two-dimensional flow, whether laminar or turbulent.

### 3. THE ENERGY SPENDING PHASE

It is expedient to consider first purely viscous (laminar) patterns, in which the mechanism is relatively simple. Indeed, in accord with Stokes,<sup>3</sup> the spending of energy has been identified with the work involved in viscous deformation. For

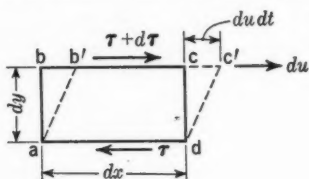


FIG. 5

two-dimensional motion (Fig. 6), when the element  $abcd$  is deformed into  $ab'c'd$ , during the time  $dt$ , the work done is the product of the shearing force  $\tau dx$  and the displacement  $du dt = \frac{du}{dy} dy dt$ , and the local unit time

rate of deformation work per unit volume is equal to

$$w_s = \tau \frac{du}{dy} \dots \dots \dots (12a)$$

or for laminar motion, with  $\tau = \mu \frac{du}{dy}$ :

$$w_s = \mu \left( \frac{du}{dy} \right)^2 \left( \frac{\text{lb} \times \text{ft}}{\text{ft}^3 \times \text{sec}} \right) \dots \dots \dots (12b)$$

Energy spending is thus concentrated in the region of high velocity gradients, which is principally near the walls of a conduit. In fact, at the axis, where  $du/dy = 0$ , the local loss is nil. The outline of the local energy spending curve  $w_s$  (curve  $o_1s_1$ , Fig. 2(b)) is thus "opposite" to that of  $w_b$ .

By analogy to Eq. 9, the cumulative spending rate—

$$\left| W_s \right|_0^{y'} = \int_0^{y'} w_s dy' \dots \dots \dots (13a)$$

—is the aggregate work spent on resistances per unit time in the shaded block of Fig. 2(a), being represented in Fig. 2(c) by curve  $O_2S_2S_2$ , in which the coordinate  $a_2s_2$  measures the area  $O_1s_1a_1$  in Fig. 2(b). For the total reach  $0-y'$ , obviously—

$$\left| W_s \right|_0^{y'o} = \int_0^{y'o} w_s dy' = \left| W_b \right|_0^{y'o} = W_o \dots \dots \dots (13b)$$

the cumulative curves  $W_b$  and  $W_s$  (Fig. 2(c)) terminating at the same point  $B_2$ . Naturally the areas  $O_1B_1A_1$  and  $O_1S_1A_1$  under curves  $w_b$  and  $w_s$  in Fig. 2(b) are equal. For any intermediary position, however, such as  $y'$ , the cumulative values  $|W_b|_0^{y'}$  and  $|W_s|_0^{y'}$  shown in Fig. 2(c) by  $a_2b_2$  and  $a_2s_2$ , do not coincide. Neither do the local values  $w_b$  and  $w_s$ , represented in Fig. 2(b) by  $a_1b_1$  and  $a_1s_1$ . Indeed in the central region of Fig. 2(b), between the axis and the intersection point  $M$ , the borrowed energy  $w_b$  exceeds the rate  $w_s$  at which energy is spent locally, whereas, nearer to the wall beyond  $M$ , the local energy rates, required for compensating the deformation work, exceed by far the comparatively small quantities  $w_b$ , available locally by borrowing. It follows, accordingly, that, between the "borrowing" and the "spending" phase, there must necessarily be an intermediate "transmittance" function, by means of which the excess of borrowed energy, available in the central region  $0-y'_M$ , is conveyed to the wall zones to cover the requirements for spending.

#### 4. THE ENERGY BALANCE EQUATION

The local energy balance can be expressed by

$$w_b = w_s + w_t \dots \dots \dots (14)$$

with  $w_t = w_b - w_s$ , designating the local "transmittance energy rate." In the central reach, with  $w_b > w_s$ ,  $w_t$  is positive, exemplifying the excess of energy subject to transmittance. In the wall region, where  $w_s$  exceeds the local supply of  $w_b$ ,  $w_t$  becomes negative, signifying that the deficiency in locally borrowed energy is supplied from the store accumulated by transmittance from the central zone.

In Fig. 2(b), the  $(B_1T_1)$ -curve is obtained by subtracting  $O_1S_1$  from  $B_1A_1$ , so that the local value for  $y'$  is  $s_1b_1 = a_1t_1$ . The resulting curve crosses the axis ( $w_t = 0$ ) at the intersection ordinate  $y'_M$ , reaching its maximum negative value  $A_1T_1$  at the solid boundary.

In Fig. 2(c), the curve  $O_2t_MA_2$  features the corresponding cumulative quantity

$$|W_t|_0^{y'} = |W_b|_0^{y'} - |W_s|_0^{y'} \dots \dots \dots (15)$$

obtained by deducting  $O_2s_2S_2$  from  $O_2b_2B_2$ . Obviously the coordinate  $a_2t_2 = a_2b_2 - a_2s_2$  measures the area  $O_1B_1b_1s_1 = O_1B_1t_1a_1$  in Fig. 2(b), indicating the aggregate excess energy, transmitted from the block toward the wall region. The maximum cumulative value  $a_{MtM}$  in Fig. 2(c), corresponding in position to the intersection point  $M$  of the local  $w_b$ -curve and  $w_s$ -curve, is the total excess energy, accumulated over the reach  $0-y'_M$  and made available through transmittance for spending in the wall zone. Obviously the coordinate  $a_{MtM}$  in Fig. 2(c) represents the positive excess area  $O_1B_1MO_1$  in Fig. 2(b), equal to the negative deficiency area  $MS_1A_1M$ . The decline of  $W_t$  beyond point  $M$  signifies the gradual consumption in the wall region of the energy accumulated by transmittance. At the solid boundary the transmittance energy exhausts its functions. Accordingly, the cumulative  $W_t$ -curve goes through zero.

## 5. THE TRANSMITTANCE FUNCTION

To grasp the "transmittance" concept consider, in two-dimensional flow, an element A (Fig. 7) in its relation to the adjacent layers B and C (for clarity the interfaces are shown separated). In its motion relative to layer B, element A exercises a positive traction, performing work on layer B at the unit time rate of  $\tau dx u$ . In turn, element A receives work from layer C at the rate  $(\tau + d\tau) dx (u + du)$ . The resulting positive work performed by the element on the adjoining fluid, over what it receives, is  $\tau dx u - (\tau + d\tau)(u + du) dx$  which, after omitting quantities of higher order of smallness, becomes

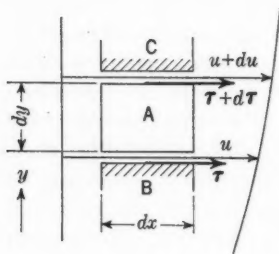


FIG. 7

$$- dx (\tau du + u d\tau) = - \frac{d}{dy} (\tau u) dy dx.$$

Dividing by the volume  $dx dy$ , the unit local rate of work transmitted to the adjacent fluid or the transmittance unit rate is

$$w_t = - \frac{d}{dy} (\tau u) = \frac{d}{dy} (-\tau u) \left( \frac{\text{lb} \times \text{ft}}{\text{ft}^3 \times \text{sec}} \right) \dots \dots \dots (16)$$

With Eqs. 11 and 12a, the local energy balance, Eq. 14, for two-dimensional laminar flow, takes the basic differential form

$$- u \frac{d\tau}{dy} = \tau \frac{du}{dy} - \frac{d}{dy} (\tau u) \dots \dots \dots (17)$$

$(w_b) \quad (w_s) \quad (w_t)$

Formally, by transposing the members, Eq. 17 may be changed into a complete differential of  $\tau u$ , which furnishes the starting point in the Stokes<sup>3</sup> analysis. However, to give Eq. 17 the proper physical significance, the members must be presented in the algebraic sequence as written. Note that, for established uniform motion, with  $\frac{d\tau}{dy}$  necessarily negative (see Eq. 10), the quantity

$- u \frac{d\tau}{dy} = w_b$  is positive. The quantity  $w_s$  is also positive, for in  $\tau \frac{du}{dy}$  both members are of the same sign (see paragraph following Eqs. 10). The only quantity that changes in sign is the local transmittance rate  $w_t$  (Eq. 16).

Since  $w_t$  is the derivative of  $-\tau u$ , the latter must feature the cumulative transmittance rate  $W_t$ . In fact, for the block  $0-y'$  in Fig. 2(a), by Eq. 16,

$$\left| W_t \right|_0^{y'} = \int_0^{y'} w_t dy' = \int_0^{y'} \frac{d}{dy'} (-\tau u) dy' = \left| -\tau u \right|_0^{y'} \dots \dots \dots (18)$$

represents the work done by the shaded block on the adjacent flow. The quantity  $\tau u$  is negative, being the product of a positive velocity and a stress, which in terms of Fig. 8 is negative. Hence  $(-\tau u)$  is positive, as indicated

by curve  $O_2t_M A_2$  in Fig. 2(c). The sign of the local value  $w_t$ , on the other hand, conditioned by the tangent to the  $W_T$ -curve, changes from positive to negative at point  $t_M$ .

As a matter of procedure, when dealing with the transmittance phase, it is expedient to start with the cumulative curve which, according to Eq. 18, is the product of the velocity curve (BA in Fig. 2(a)) and the stress line (OC in Fig. 2(a)). The maximum value of  $-u\tau$  is reached at the intersection point M in Fig. 2(b) corresponding to  $w_t = w_b - w_s = 0$ . Point M is to be referred to as the "reversal point."

#### Cumulative Energy Balance Equation.—

The local equation (Eq. 17) is complemented for two-dimensional uniform flow by the cumulative expression

$$-\int_{u_1}^{u_2} u \frac{d\tau}{dy} dy = \int_{u_1}^{u_2} \tau \frac{du}{dy} dy - \int_{u_1}^{u_2} \frac{d}{dy} (\tau u) dy \dots \dots \dots (19)$$

or, in relation to Fig. 2:

$$\int_0^{y'} \left( -u \frac{d\tau}{dy} \right) dy' = \int_0^{y'} \tau \frac{du}{dy} dy' + \left| -\tau u \right|_0^{y'} \dots \dots \dots (20)$$

and

$$\gamma S \left| q \right|_0^{y'} = \int_0^{y'} w_s dy' + \left| -\tau u \right|_0^{y'} \dots \dots \dots (21)$$

all the members (see Fig. 2(c)) being positive.

Again, as in the case of Eq. 11 and Eq. 9, no restrictions were imposed on the nature of the stress-forming agency. The "transmittance" phase formulas, Eqs. 16 and 18, apply to all forms of flow, laminar or turbulent.

## 6. MECHANICAL ANALOGY

The essence of the "transmittance function" may be demonstrated by a mechanical model. In Fig. 9 a sequence of wheels is mounted on a common frame, with adequate friction between the contacting surfaces. Because of this friction, energy derived from the "driving" agencies  $P_1, P_2, \dots$ , is transmitted to where it is spent on the resistances  $R$ . Slipping between successive wheels will mean that, although the friction components such as  $|T_{p1}|_{1,2} = P_1 \frac{r_d}{r_o}$  and  $|T_{p1}|_{2,1}$  (Fig. 9(b)) are equal, the velocities  $u_1$  and  $u_2$  differ, and thus only a part of the power  $|T_{p1}|u_1$  derived from  $P_1$  (namely,  $|T_{p1}|u_2$ ) will be transmitted to the next wheel. The difference

$$|T_{p1}|u_1 - |T_{p1}|u_2 = |T_{p1}|\Delta u_{1,2} \dots \dots \dots (22)$$

represents a local mechanical "energy loss," an amount dissipated directly

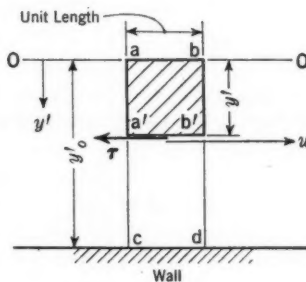


FIG. 8

into molecular heat. The work done cumulatively by all the preceding driving agencies ( $P_1$  to  $P_i$ ) through wheel  $i$  is

$$F_i u_i = u_i \sum_1^i |T_p| \dots \dots \dots (23)$$

Of this work, however, only  $F_i u_i$  is passed to the next wheel ( $j$ ), the amount  $F_i (u_i - u_j)$  being dissipated locally in the slip between ( $i$ ) and ( $j$ ). As a

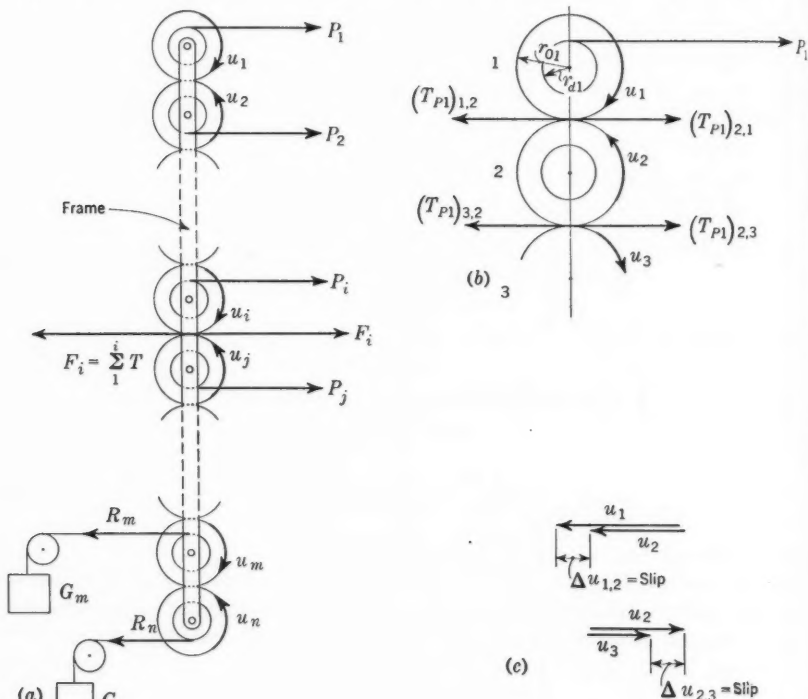


FIG. 9

result, the "cumulative" energy balance for the wheel sequence is expressed by

$$\sum_1^i |T_p| u = \sum_1^i (F \Delta u) + u_i F_i \dots \dots \dots (24)$$

Eq. 24 is analogous to Eq. 21. The friction resultant  $F_i$  is the simile of the stress  $\tau$  in Fig. 8;  $\sum_1^i |T_p| u$  is equivalent to the borrowed energy  $|W_b|_0^v$ ; the local "slip" loss (Eq. 22) compares with  $w_s$  in Eq. 12a; and  $\sum (F \Delta u)$  in Eq. 24 is the analogue of  $|W_s|_0^v$ . Finally, the "transmittance" member  $u_i F_i$  is the equivalent, both in meaning and form, of the product  $|\tau u| = |W_t|_0^v$ .

## 7. ENERGY "SPENDING" AND "DISSIPATION"

Loss of energy signifies ultimate dissipation into molecular heat. Energy borrowed from the flow changes its mechanical aspects and is transformed into thermal form. Such transformation is caused finally by viscous attrition, which enhances intermolecular activity. The ultimate dissipation develops thus on an ultramicroscopic molecular scale.

In laminar patterns the process of dissipation is assumed to coincide with viscous deformation (see Section 3). Mechanical energy spent in deformation is dissipated through transformation into heat in loco.

In turbulent flow, "spending" and "dissipation" coincide only partly. The term "spending" will be used to signify only the idea that energy borrowed from the flow has lost, irretrievably, its original flow form; and "dissipation," under all circumstances, will designate the ultimate "degradation" of mechanical energy into heat.

## 8. EXAMPLES OF ENERGY BALANCES IN LAMINAR MOTION

In some of the simpler laminar patterns, the stress and velocity quantities can be expressed in definite mathematical form, permitting an appraisal of the energy exchange process by purely analytical procedure.

(a) *Two-Dimensional Motion, Between Parallel Plates (Fig. 2).*—With  $q_o = U y'_o$  as the unit width discharge between the axis and the wall, the local velocity is

$$u = u_M \left[ 1 - \left( \frac{y'}{y'_o} \right)^2 \right] \dots \dots \dots (25)$$

in which  $u_M = \frac{3}{2} U$ . Furthermore,

$$-\frac{dp}{dx} = \frac{3 \mu U}{(y'_o)^2} \dots \dots \dots (26a)$$

and

$$\tau_o = -\frac{3 \mu U}{y'_o} \dots \dots \dots (26b)$$

Accordingly in Eqs. 7 and 8,

$$W_o = \frac{\tau_o^2 y'_o}{3 \mu} \dots \dots \dots (27a)$$

and

$$w_o = \frac{\tau_o^2}{3 \mu} \dots \dots \dots (27b)$$

By elementary algebraic procedure (which is omitted), one arrives at the following dimensionless expressions for the local and cumulative unit rates that appear in Eqs. 17 and 20:

$$w_b = \frac{3}{2} w_o \left[ 1 - \left( \frac{y'}{y'_o} \right)^2 \right]; \quad w_s = 3 w_o \left( \frac{y'}{y'_o} \right)^2; \\ w_t = \frac{3}{2} w_o \left[ 1 - 3 \left( \frac{y'}{y'_o} \right)^2 \right] \dots \dots \dots (28a)$$

and

$$\begin{aligned} |W_b|_0^{v'} &= \frac{3}{2} W_o \left[ \frac{y'}{y'_o} - \frac{1}{3} \left( \frac{y'}{y'_o} \right)^3 \right]; & |W_s|_0^{v'} &= W_o \left( \frac{y'}{y'_o} \right)^3; \\ |W_t|_0^{v'} &= \frac{3}{2} W_o \left[ \frac{y'}{y'_o} - \left( \frac{y'}{y'_o} \right)^3 \right] \dots \dots \dots (28b) \end{aligned}$$

The curves of these equations are plotted in Fig. 2.

(b) *Cylindrical Pipe*.—The Poiseuille relation,

$$-\frac{dp}{dx} = \frac{8 U \mu}{r_o^2} \dots \dots \dots (29)$$

with  $R = \frac{r_o}{2}$  and  $u_m = 2 U$ , leads to the following expressions:

$$\tau_o = -\frac{4 \mu U}{r_o} \dots \dots \dots (30a)$$

$$u = -\frac{\tau_o r_o}{2 \mu} \left[ 1 - \left( \frac{r}{r_o} \right)^2 \right] \dots \dots \dots (30b)$$

$$W_o = \frac{\pi r_o^2 \tau_o^2}{2 \mu} \dots \dots \dots (30c)$$

and

$$w_o = \frac{\tau_o^2}{2 \mu} \dots \dots \dots (30d)$$

In establishing the local rates, it is expedient to operate with an elementary cylindrical shell. For example, the work involved in the deformation shown in Fig. 10(a) results in the formula:

$$w_s = \tau \frac{du}{dr} \dots \dots \dots (31)$$

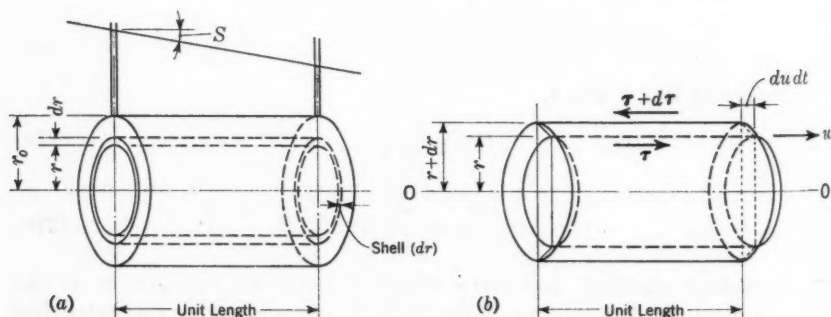


FIG. 10

The cumulative quantities should refer to an internal cylinder of radius  $r$  and unit length. The final dimensionless interrelationships are:

$$w_b = 2 w_o \left[ 1 - \left( \frac{r}{r_o} \right)^2 \right]; \quad w_s = 2 w_o \left( \frac{r}{r_o} \right)^2; \quad w_t = 2 w_o \left[ 1 - 2 \left( \frac{r}{r_o} \right)^2 \right] \dots (32)$$

and

$$\begin{aligned} |W_b|_0^r &= 2 W_o \left( \frac{r}{r_o} \right)^2 \left[ 1 - \frac{1}{2} \left( \frac{r}{r_o} \right)^2 \right]; & |W_s|_0^r &= W_o \left( \frac{r}{r_o} \right)^4; \\ |W_t|_0^r &= 2 W_o \left( \frac{r}{r_o} \right)^2 \left[ 1 - \left( \frac{r}{r_o} \right)^2 \right] \dots\dots\dots (33) \end{aligned}$$

The cumulative curves are reproduced for comparison, with the eventual turbulent outlines, in Fig. 11.

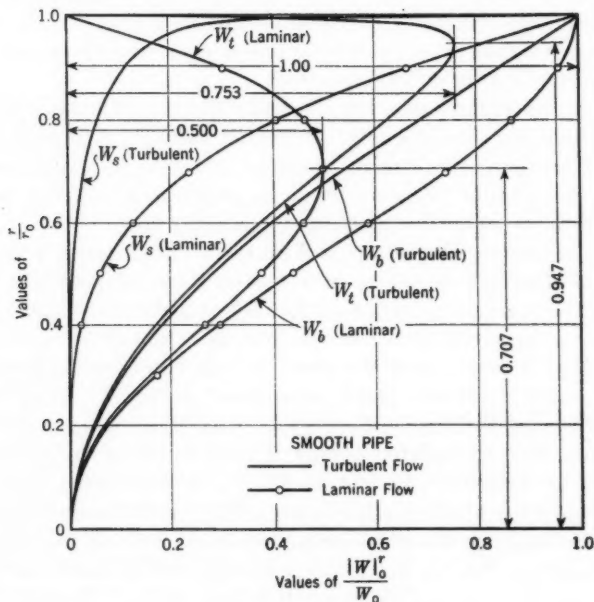


FIG. 11

### 9. TURBULENT FLOW PATTERNS

Since no restrictions were introduced to limit the expressions for the borrowing rate  $w_b$  (Eq. 11) and the transmittance rate  $w_t$  (Eq. 16), the equations are directly applicable to turbulent flow. Naturally,  $\bar{\tau}$  will be the local Reynolds stress, and  $\bar{u}$  the mean steady local velocity, whereas the quantities

$\bar{w}_b = -u \frac{d\bar{\tau}}{dy}$  and  $\bar{w}_t = \frac{d}{dy} (-\bar{\tau} \bar{u})$  in Eq. 18 and Eq. 21 will feature the temporal average unit borrowing and transmittance rates. Because of the complex nature of turbulent motion, the spending rate  $w_s$  cannot be appraised a priori as in the case of Eq. 12b for laminar flow. On the other hand, from the standpoint of a complete differential, the mean local spending rate for two-dimensional turbulent flow—

$$\bar{w}_s = \bar{w}_b - \bar{w}_t = -u \frac{d\bar{\tau}}{dy} - \frac{d}{dy} (-\bar{\tau} \bar{u}) = \bar{\tau} \frac{d\bar{u}}{dy} \dots\dots\dots (34)$$

—must conserve the mathematical form of Eq. 12a. The  $w_t$ -quantity indicates the unit rate at which flow energy disappears under the action of agencies pertinent to the turbulent process.

Furthermore, the incumbent stress and velocity quantities cannot be derived from analytical expression as in Section 3, and must be taken directly from observation. Empirical data vary widely, depending on wall roughness, the Reynolds number, and other circumstances of the flow. Nevertheless, energy balances for turbulent flow display certain consistent characteristics, which happen to differ substantially from laminar outlines, and reflect the essence of turbulence dynamics. A typical example is given in Fig. 1, based on Göttingen data<sup>8</sup> relating to turbulent flow in a 10 cm smooth pipe, with  $U = 876$  cm per sec, and  $-dp/dx = 480$  g per cm<sup>3</sup>.

By using in Eq. 34 the values for  $\bar{\tau}$ ,  $\bar{u}$ , and  $d\bar{u}/dy$  contained in the Nikuradse report, one computes the consecutive local unit rates  $w_b$ ,  $w_t$ , and  $w_s$ , as plotted in Fig. 1. The cumulative curves, obtained from the local outlines through step-by-step integration are presented, in dimensionless form, in Fig. 11, together with the comparative plottings for laminar flow.

The striking feature, characteristic of all turbulent patterns, is the intensive concentration of energy spending in a narrow zone next to the wall. For laminar flow the reversal point (M) is located at a distance  $0.293 r_o$  from the wall, whereas in turbulent motion it is shifted to within  $0.053 r_o$  from the solid boundary. Furthermore, over the reach  $0-r_M$ , constituting about 95% of the radial distance, relatively small amounts of energy are spent locally, the major part of the locally borrowed energy  $w_b$  being transmitted to the wall zone ( $r_M-r_o$ ) where the accumulated energy is used to offset the unusually high requirements for spending. In fact, in Fig. 1, the calculated local values of  $w$ , and  $w_t$  close to the solid boundary outreach by far the bounds of the figure. Accordingly, to appraise the phenomena, one must refer to the dimensionless cumulative diagram, Fig. 11, in which the specific characteristics of the energy exchange in turbulent motion become particularly lucid. Through the central region, the cumulative transmittance curve  $|W_t|_0^r$  follows closely the borrowing outline  $|W_b|_0^r$ . In fact, over the reach  $0-M$ , in which more than 90% of the total energy  $W_o$  is withdrawn (borrowed) from the flow, the cumulative loss is less than 17%. By contrast over 83% of  $W_o$  is spent in the narrow ( $0.05 r_o$ ) wall zone, where the quite insignificant local borrowing accounts for less than 8% of  $W_o$ . In other words, the concentrated "spending" in the wall zone takes effect principally at the expense of the transmitted energy.

One should infer, from the shape of the turbulent curves, that the energy exchange in the central zone and in the border zone must be actuated by entirely different physical factors. Conventionally, in fact, one may accept the premise that the "reversal radius"  $r_M$  separates the fluid body into two parts, of which the central cylindrical block ( $0-r_M$ ) will be called the "transmittance region," whereas the annular space near the wall ( $r_M-r_o$ ) will be referred to as the "conversion zone" (see Section 12).

<sup>8</sup> "Gesetzmässigkeiten der turbulenten Strömung in Glatten Röhren," by J. Nikuradse, *Forschungsheft*, Verein Deutscher Ingenieure, No. 356, 1932, pp. 20 and 31.

## 10. THE MECHANICS OF TURBULENT FRICTION LOSS

Turbulent motion is characterized by the presence, in the flow, of an array of eddies that "swarm" in an apparently unpredictable manner and at each instant add their rotary components to the respective local velocities of the main motion. The ever-fluctuating turbulent pattern, superimposed over the steady flow structure, devolves thus from the composite action of a multitude of individual vorticing units.

The energy of a fluid in turbulent motion is compounded from the energy of the main flow, characterized locally by the energy head

$$e = z + \frac{p}{\gamma} + \frac{u^2}{2g} \dots \dots \dots (35)$$

and by the kinetic energy of the superimposed turbulent structure. The latter, being essentially the composite energy of the swarming eddies, is defined locally by value of  $\rho \overline{(u')^2}/2$ , where

$$\overline{(u')^2} = \overline{(u')^2} + \overline{(v')^2} + \overline{(w')^2} \dots \dots \dots (36)$$

is the temporal mean square of the local turbulent fluctuations in the three coordinate directions.

A statistically permanent state of turbulence implies the maintenance of a steady average eddying structure, which means that new vorticing units must be generated incessantly to replace the eddies "extinguished" by attrition in the course of their swarming motion through the flow. The generation of eddies infers that they are endowed with energy, which must necessarily be supplied from the energy store of the main flow. Indeed, the primary characteristic of the turbulence mechanism is that energy, at an apportioned rate, is withdrawn continuously from the main flow, and is "invested" into turbulent eddies, such "investment" involving the conversion of the borrowed flow energy into vortical eddying form. The generation of turbulence is thus synonymous with the producing of the turbulent eddies. Turbulent energy, manifest in turbulent fluctuations, is originally flow energy, transformed into vortical form. By its very nature the conversion is an irreversible process, because energy, once converted into eddies, can by no means be restored to the flow. For such reason, so far as the main flow is concerned, when energy withdrawn from the latter is "spent" in the creation of turbulent eddies, it may be considered "lost." Actually the "lost" flow energy reappears, at least partially, in the converted vortical form, which animates the turbulent structure. The final dissipation of such energy takes place after conversion, and at the time when, in the course of swarming, the mechanical energy with which the eddies were initially endowed is transformed (degraded) through viscous attrition into molecular heat.

According to prevailing views, the generation of turbulent eddies in established "friction" flow is largely concentrated in a narrow zone near the solid boundaries of a conduit. L. Prandtl<sup>9</sup> most appropriately qualifies his definition

<sup>9</sup> *Proceedings, 5th International Cong. for Applied Mechanics, Cambridge, Mass., John Wiley & Sons, Inc., New York, N. Y., 1939, p. 367.*

of such boundary zones as "the actual eddy producing mill," whereas the "spreading" of the eddies toward the central region is said to take place "rather passively." Indeed, the faculty of generating eddies seems to be a specific property, inherent to discontinuity "vortex sheets" that are subjected to excessive strain. Whenever they are overstrained, such vortex sheets, or layers of concentrated vorticity, become unstable; and after passing through a cycle of ever-increasing self-induced pendulations, they terminate by eventually "curling," or being "rolled" up, into "chains" of eddies or a sequence of intermittent lumps of condensed vorticity. The process is comparatively well defined in the case of "free" discontinuity sheets caused in the interior of a fluid by "separation." As illustrated by Fig. 12, the formation of the eddies, in

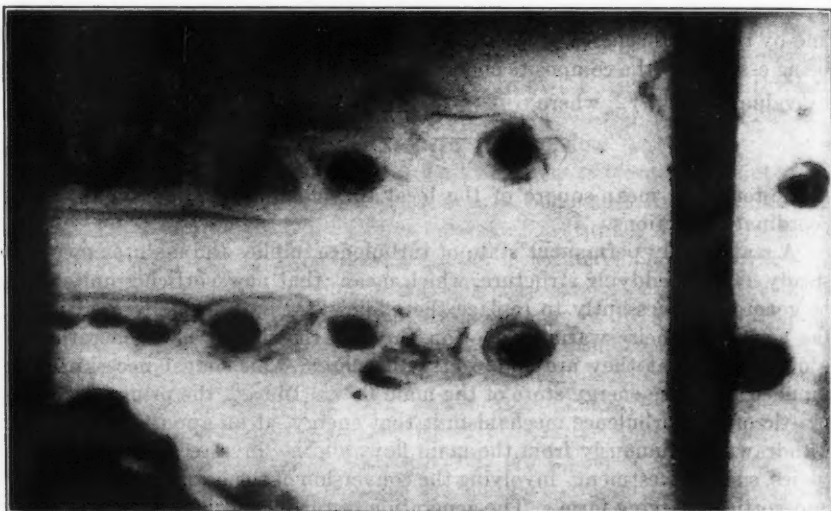


FIG 12.—THE "ROLLING UP" OF ANNULAR EDDIES, ON THE SURFACE OF A CONTRACTED VEIN ENTERING A PIPE

such instances, may be observed visually. (In these observations by L. Schiller and his associates,<sup>10</sup> the original vortex sheet was made visible by the injection of dye.) In friction flow, past solid surfaces, the reach of excessive strain is supposedly associated with the boundary zone near the wall where, over an exceedingly thin "sublayer," the velocity drops abruptly to zero. Thus far, existing techniques have not permitted the direct observation of turbulence generation near solid boundaries. There is ample indirect evidence, nevertheless, pointing to the essential similarity of the case with that of "free" separation sheets. (For further information in this generally still obscure and mysterious realm, the reader is referred to the general literature in this field.)<sup>11,12</sup>

<sup>10</sup> *Proceedings, 5th International Cong. for Applied Mechanics, Cambridge, Mass., John Wiley & Sons, Inc., New York, N. Y., 1939, p. 315.*

<sup>11</sup> "Fundamentals of Hydro- and Aeromechanics," based on lectures of L. Prandtl, by O. G. Tietjens, translated by L. Rosenhead, 1st Ed., McGraw-Hill Book Co., Inc., New York and London, 1934, p. 222.

<sup>12</sup> "The Formation of Vortices from a Surface of Discontinuity," by L. Rosenhead, *Proceedings, Royal Soc. of London, Series A, Vol. CXXXIV, 1931-1932, pp. 170-192.*

The pivotal phase in the sequence of events is conversion, in the course of which turbulent eddies are generated incessantly in the conversion zone. The preconversion phases (Section 9) deal with energy of the main flow. As illustrated by Fig. 11, flow energy is borrowed largely in the middle regions of a cross section and through transmittance is concentrated near the wall, being made available for conversion. In the post-conversion phase, the remainder of the originally borrowed mechanical energy is in turbulent eddying form. Individual vorticing units, assumedly of the Fig. 12 type, are "cast off" or "shed" from the generating zone into the adjacent flow, and as the eddies spread across the conduit on generally oblique paths, their initial energy is gradually dissipated into molecular heat. The possible interreaction between a vortical unit and the surrounding

flow is illustrated by Fig. 13. On the top of the eddy, the rotary component  $u_t$  adds to the local velocity  $u_1$ , whereas at the bottom it subtracts from  $u_2$ . The resulting difference in velocity heads results in a transverse pressure differential, prompting the eddy to move in a crosswise direction away from the wall. The "shedding" of the eddies from the seat where they are originally generated may be thus considered as caused by a "Magnus effect," similar in nature to the "lift" of an airfoil or to the propelling force in a rotor ship. The transverse motion, prompted by such induced pressure forces, combines with the drag of the current which tends to wash the eddy downstream.

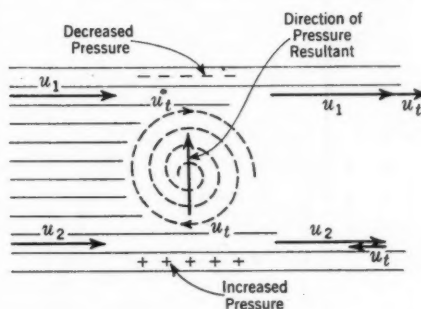


FIG. 13

The transformation of the mechanical vortical energy into thermal state is by viscous attrition. Indeed, the local dissipation rate may be expressed in terms of the Taylor formula<sup>13,14</sup>

$$w_d = \kappa \mu \frac{(\overline{u'_t})^2}{\lambda^2} \dots \dots \dots (37)$$

in which  $\kappa$  is a numerical coefficient, and  $\lambda$  is a characteristic dimension of the small-size eddying group in the vorticing composite. In an eddying composite, energy is dissipated in large part through the instrumentality of the smallest size eddies, or the so-called micro-turbulence fringe. Large eddies, in turn, are responsible for producing the turbulent stress and more generally for actuating convective transfer. The large eddies, accordingly, enhance turbulent diffusion, and their direct contribution to dissipation is comparatively small. It is surmised that, in the course of swarming, the larger eddies are subject to a process of "grinding," by which they are split continuously into smaller

<sup>13</sup> "Statistical Theory of Turbulence," by G. I. Taylor, *Proceedings, Royal Soc. of London, Series A*, Vol. CLI, 1935, pp. 430-454.

<sup>14</sup> "Modern Developments in Fluid Dynamics," edited by S. Goldstein, Clarendon Press, Oxford, England, 1938, Vol. II, p. 221.

units. G. I. Taylor has treated this general subject and has presented a bibliography.<sup>15,16</sup>

The sequence involved in the mechanism of turbulent friction loss is presented schematically in Fig. 14. Of the total borrowed flow energy  $W_0$ , the amount  $W_c$  is made available in the conversion zone. Energy  $W_e$  is less than

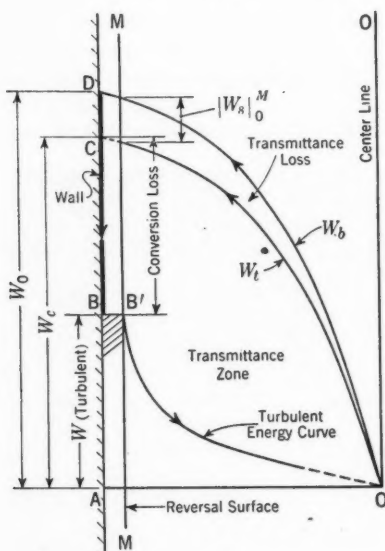


Fig. 14

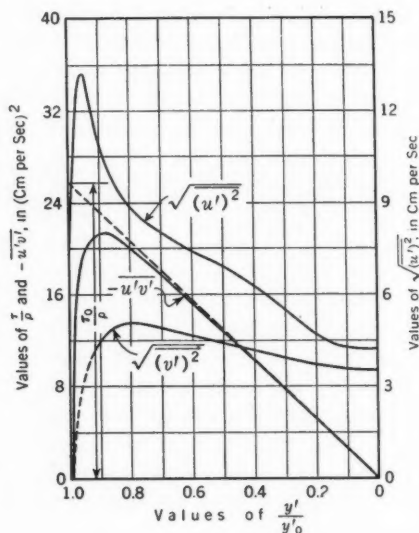


Fig. 15

$W_0$  by about the cumulative loss  $|W_e|_0^M$  in the transmittance zone. In the absence of adequate experimental knowledge, one cannot judge as to what part of  $W_c$  is dissipated locally in the course of conversion. No doubt, however (see Section 12), the conversion process is accompanied by a considerable inherent loss. The latter, in Fig. 14, is symbolized by the ordinate CB. The remaining  $W$  (turbulent) = BA represents the part of the original  $W_0$  which survives conversion and which, liberated in eddying form, furnishes the "animating" source by which the state of turbulence, as revealed in its outward manifestations, is actuated and maintained. Finally, curve B'-O portrays the gradual extinction of the eddying structure or the transverse "decay of turbulence," by the dissipation of mechanical energy into molecular heat (Eq. 37).

An exact quantitative appraisal of the sequence would require experimental data of the type presented in Fig. 11 complemented by detailed records showing reliably, in three-dimensional fashion, the distribution of turbulence intensity across the cross section. No complete data of this nature are available. Fig. 15 shows, in part, the distribution of the axial and transversal turbulent com-

<sup>15</sup> "Some Recent Developments in the Study of Turbulence," by G. I. Taylor, *Proceedings, 5th International Cong. for Applied Mechanics*, Cambridge, Mass., John Wiley & Sons, Inc., New York, N. Y., 1939, p. 301.

<sup>16</sup> "Turbulence Investigations at the National Bureau of Standards," by Hugh L. Dryden, *ibid.*, p. 365.

ponents in a rectangular conduit.<sup>17</sup> As seen, the axial intensity of turbulence, measured by the root mean square  $\sqrt{(u')^2}$ , reaches a sharp maximum in the immediately proximity of the wall, readily corresponding to the outward boundary of the conversion region. From there on, the intensity recedes rapidly toward the center of the conduit.

### 11. EFFECT OF WALL ROUGHNESS

A basic premise underlying the Prandtl-von Kármán turbulence theory is the organic link between the stress structure and the shape of the velocity curves.<sup>18</sup> Experiments reveal that, if the roughness of the walls of a given

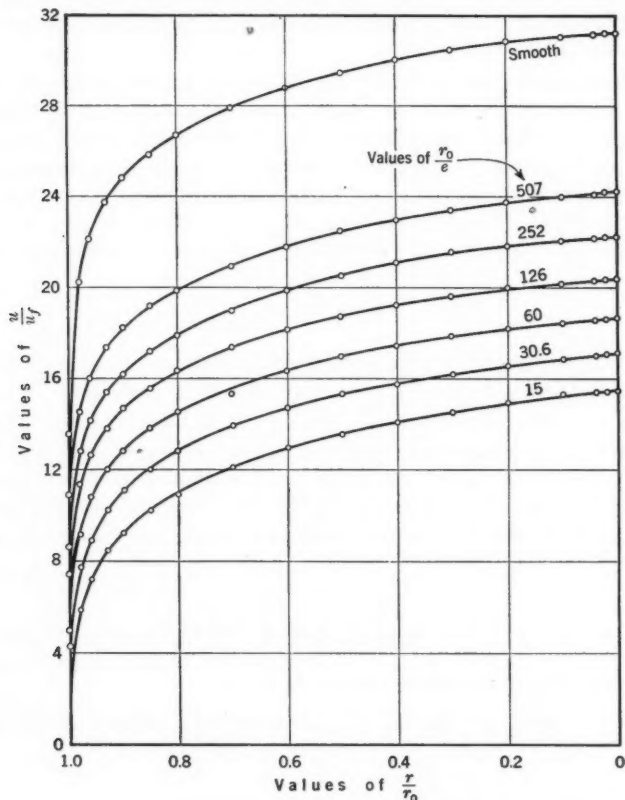


FIG. 16

conduit of radius  $r_0$  is modified consecutively, the flow  $Q$  being adjusted to yield in each instance the same resistance slope  $S$ , and hence the same stress structure (Eqs. 3 and 4), then the respective velocity profiles in the central "turbulent"

<sup>17</sup> "Beitrag zum Turbulenz Symposium," by L. Prandtl, *Proceedings, 5th International Cong. for Applied Mechanics*, Cambridge, Mass., John Wiley & Sons, Inc., New York, N. Y., 1939, p. 345.

<sup>18</sup> "The Mechanics of Turbulent Flow," by B. A. Bakhmeteff, Princeton Univ. Press, Princeton, N. J., 1936, p. 55.

zone will follow identically shaped parallel outlines (see Fig. 16). In fact, if the observed velocities were plotted from a common maximum point, the curves for the different roughnesses would coincide over the greater part of the cross section, the eventual differences becoming apparent only in the comparatively narrow boundary zones.<sup>19</sup>

In regard to the energy balances, an unvarying resistance slope makes the total energy loss  $W_o = \gamma Q S$  proportional to the respective average velocity. Furthermore, congruent velocity profiles signify that the velocity gradients  $du/dy$  in the central turbulent zone are the same for all degrees of roughness, and that therefore in such a zone the local losses  $w_s$  by Eq. 34 will be identical in all instances. The same obviously will pertain to the cumulative values  $|W_s|_0^r$ , as long as one does not transgress to a point where the velocity profiles cease to be congruent.

The inescapable conclusion is that the spending of energy in the transmittance region, the latitude of which may be broadly said to coincide with the central "turbulent" zone, is solely dependent on the stress structure, and that the spending process in the transmittance zone is implemented by the same agencies that shape the stress patterns (see Section 12). Deducting the common loss in the transmittance zone  $|W_s|_0^M$  from the respective  $W_o$ -quantities (which are proportional to the varying bulk velocities), one obtains the energy quantities

$$W_c = W_o - |W_s|_0^M = \gamma S A U - |W_s|_0^M \dots \dots \dots (38)$$

which for one or another degree of wall roughness are to be spent in the conversion (boundary) zones. Paradoxically, the largest value of  $W_c$  occurs in the case of a smooth surface. The rougher the walls, the smaller the value of  $W_o$  and therefore of  $W_c$ , always assuming an unvarying resistance slope.

The relation between the velocities and the wall roughness is summarized in terms of dimensionless ratios in Fig. 16.<sup>20,21</sup> Velocity profiles for pipes of different size and different relative roughness are plotted versus  $r/r_o$  in terms of the so-called "friction velocity"

$$u_f = \sqrt{\frac{\tau_o}{\rho}} = U \sqrt{\frac{c_f}{2}} = U \sqrt{\frac{f}{8}} \dots \dots \dots (39)$$

$c_f$  is the conventional turbulent friction factor in

$$\tau_o = \rho c_f \frac{U^2}{2} \dots \dots \dots (40a)$$

and  $f$  is the coefficient of friction in the pipe formula

$$S = \frac{f}{D} \frac{U^2}{2g} \dots \dots \dots (40b)$$

<sup>19</sup> "The Mechanics of Turbulent Flow," by B. A. Bakhmeteff, Princeton Univ. Press, Princeton, N. J., 1936, p. 56.

<sup>20</sup> "Strömungsgesetze in rauen Rohren," by J. Nikuradse, *Forschungsheft*, Verein Deutscher Ingenieure, No. 361, 1933, p. 8.

<sup>21</sup> "The Mechanics of Turbulent Flow," by B. A. Bakhmeteff, Princeton Univ. Press., Princeton, N. J., 1936, p. 97.

(Fig. 16 refers to flow in pipes ranging from 1 in. to 4 in. in diameter,  $D$ , the walls of which were roughened artificially by uniform sand grains of different sizes. The inverse ratios of relative roughness  $r_o/e$  are the quotients of the radius of the pipe by the grain size. The shape of the roughness grains is assumed to be "similar.")

In the central reaches the velocity profiles in Fig. 16 show congruence in outline, which is to be anticipated, since reducing the velocities to the term  $u_f$  is equivalent to referring the velocity pattern to a common stress structure. Since the curves are dimensionless and imply no restrictions with regard to the absolute dimensions of the conduit or the size of the roughness grains, the diagram typifies universal velocity patterns for the "standard" Göttingen sand roughness type. Accordingly the experimental data, underlying Fig. 16 and explicitly tabulated in the Göttingen Report,<sup>22</sup> may be used for disclosing, in equally generalized manner, the comparative features of the internal energy exchanges for pipes of the respective relative wall roughnesses. In fact, in terms of the "friction" velocity (Eq. 39) the pressure gradient is

$$-\frac{dp}{dx} = \gamma S = \left| \frac{2 \tau_o}{r_o} \right| = \frac{2 \rho u_f^2}{r_o} \dots \dots \dots (41)^*$$

and the total energy loss and the average spending rate are

$$W_o = 2 \pi r_o |\tau_o| U = 2 \pi r_o \rho u_f^3 \frac{U}{u_f} \dots \dots \dots (42a)$$

and

$$w_o = \frac{W_o}{\pi r_o^2} = \frac{2 \rho u_f^3}{r_o} \frac{U}{u_f} \dots \dots \dots (42b)$$

The analysis may be confined to the cumulative quantities only. In expressing borrowed energy, replace the pressure gradient in  $w_b$  (Eq. 11) by Eq. 41; substitute  $u_f \frac{u}{u_f}$  for  $u$ ; and  $r_o \frac{r}{r_o}$  for  $r$ . Then:

$$\left| W_b \right|_0^r = -\frac{dp}{dx} \int_0^r 2 \pi r u dr = 2 \pi r_o \rho u_f^3 \int_0^{r/r_o} 2 \frac{r}{r_o} \frac{u}{u_f} d \left( \frac{r}{r_o} \right) \dots (43a)$$

The subintegral quantity is twice the static moment, with regard to the axis, of the area bounded by the velocity curve  $u/u_f$  in Fig. 16, between the limits 0 and  $r/r_o$ .

The cumulative transmittance rate, for a cylinder of radius  $r$ , takes the form

$$\left| W_t \right|_0^r = 2 \pi r (-\tau u)_0^r = 2 \pi r_o \rho u_f^3 \left[ \left( \frac{r}{r_o} \right)^2 \frac{u}{u_f} \right] \dots \dots \dots (43b)$$

in which the quantity in brackets equals the product of the squared relative distance ( $r/r_o$ ) in Fig. 16 by the corresponding  $u/u_f$ .

After the curves for the borrowing and for the transmittance phases are computed and plotted, the cumulative spending rate

$$\left| W_s \right|_0^r = \left| W_b \right|_0^r - \left| W_t \right|_0^r \dots \dots \dots (44)$$

<sup>22</sup>"Strömungsgesetze in rauen Röhren," by J. Nikuradse, *Forschungsheft, Verein Deutscher Ingenieure*, No. 361, 1933, p. 12.

is determined by subtracting the outline for  $W_t$  from that of  $W_b$ . Eqs. 43a and 43b contain the quantity  $2 \pi r_o \rho u_f^3$  which, with Eq. 42a, becomes equal to  $W_o \frac{u_f}{U}$ ; and this value is used as a reference basis in plotting the curves. The resulting diagrams (Fig. 17), of which Fig. 17(b) is reduced to dimensionless

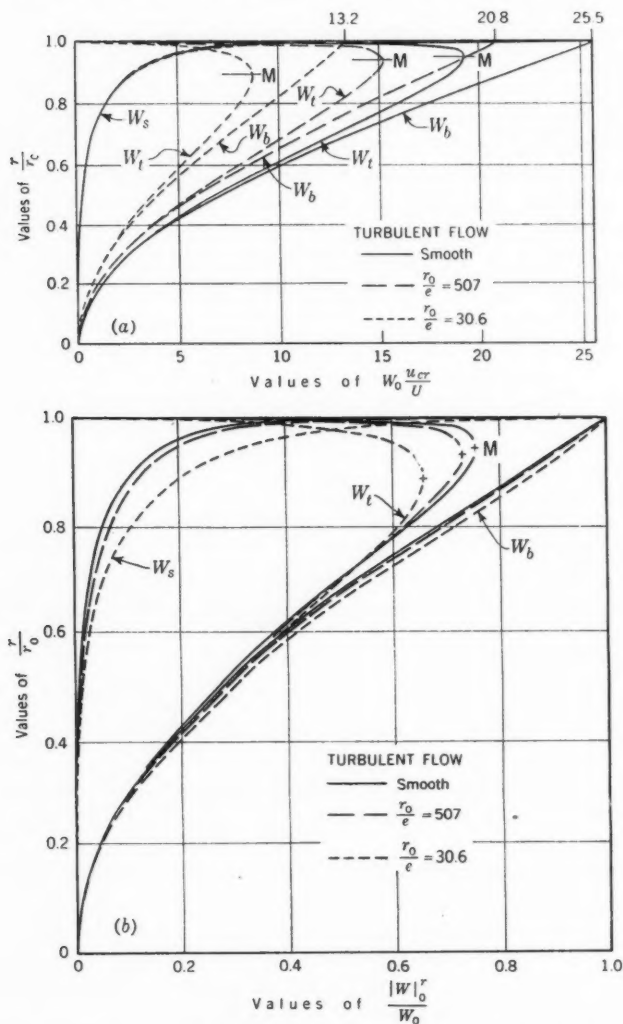


FIG. 17.—CUMULATIVE ENERGY BALANCE IN TURBULENT FLOW IN PIPES

terms, are most instructive. The reversal M point is closest to the boundary in the instance of a smooth wall. As the roughness increases, the transverse latitude of the conversion zone also grows; in the extreme case for which

$r_o/e = 30.6$ , it is about 10% of  $r_o$ . As expected, within the transmittance zones, the cumulative loss is shown by a single outline. In fact,  $W_s$  shows no appreciable discrepancies, up to the closest observable range ( $r = 0.98 r_o$ ) from the wall, which means that conversion is actually concentrated in a very narrow layer, the thickness of which is only a small portion of the reach  $r_M - r_o$ . In the case of the smooth pipe, the energy destined for conversion constitutes nineteen

$\left(W_o \frac{u_f}{U}\right)$ -units of a total of 25.5, or about 75%. For the roughest surface, the corresponding energy is 8.75 units of a total of 13.2, or 66%. On the whole, however, the dimensionless curves of Fig. 17 are a rather compact set, suggesting that, over a great variety of circumstances, energy exchange patterns in turbulent flow exhibit generically similar characteristics.

## 12. THE NATURE OF THE ENERGY SPENDING PROCESS

In treating the factors that shape the stress structure in turbulent patterns, it is customary to distinguish between the "viscous" and the "turbulent" agencies. Thus in

$$\tau = \mu \frac{d\bar{u}}{dy} - \left| \rho l^2 \left( \frac{d\bar{u}}{dy} \right)^2 \right| \dots \dots \dots (45).$$

the quantity between the vertical bars represents (according to Prandtl) the effect of the Reynolds "momentum transfer," produced by transverse turbulent mixing.<sup>23</sup> One could expect that the same agencies would be instrumental in determining the manner in which energy, withdrawn from the flow, is "spent" in the different regions of a cross section.

*Conversion Zone.*—Paradoxically the function of conversion, which was said to be synonymous with the generation of turbulence (Section 10) reverts primarily to viscous action. The "casting off" of eddies appears to be the ultimate phase in a sequence of cyclic transformations suffered by viscous vortex sheets subjected to excessive strain. The intensity of the latter is affected by the fact that, in the immediate vicinity of the wall, the boundary stress by Eq. 40a, corresponding in magnitude to the general turbulent structure, must be "carried across" the laminar sublayer by viscous shear. This results in an extremely sharp velocity gradient

$$\left( \frac{du}{dy} \right)_{y \rightarrow 0} = \left| \frac{\tau_o}{\mu} \right| = \frac{\rho u_f^2}{\mu} = \frac{c_f U^2}{\nu} \dots \dots \dots (46)$$

which exceeds by hundreds, if not by thousands, the ordinary "laminar" strain underlying the appraisal of deformation work in Fig. 6. The extreme character of the ensuing deformation is suggested by Fig. 18(a); Fig. 18(b) shows in turn the ultimate phase of the cyclic transformation, with the original viscous sheet rolled up into interspaced eddies, ready to be detached and "cast off" into the main flow.<sup>11,12</sup>

The deformation work involved in the initial straining of the vortex sheet can be appraised by means of Eqs. 12. Indeed, it is viscous deformation work,

<sup>23</sup> "The Mechanics of Turbulent Flow," by B. A. Bakhmeteff, Princeton Univ. Press, Princeton, N. J., 1936, p. 44.

in the first instance, which absorbs the energy made available through transmittance and is destined for conversion.

In the case of laminar motion, as stated in Section 7, all the deformation energy is dissipated in loco. In the generation of turbulence, on the other hand,

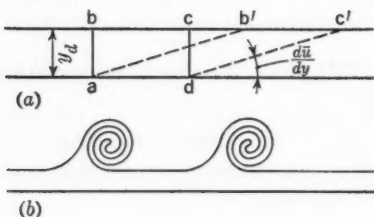


FIG. 18

part of the energy absorbed in deformation reappears in converted eddying form. Such an apparent difference in behavior can be explained at present only by hypothetical conjecture. The answer may lie in the dimensional scales of the respective phenomena. In laminar motion the action of viscosity, when one layer slides past another, is on a molecular scale. Any eddies

formed in the interface between layers, which constitute the vortex sheets, would be of ultramicroscopic size and therefore (Eq. 37) would become subject to a most intensive dissipative action. As a result the tiny "molecular" eddies become extinguished in loco without having a chance of breaking away as individual units.

By contrast in the boundary zone of Fig. 18, the pendulations, induced by the instability of the initial sheet, result in concentrated lumps of vorticity of a molar, macroscopic size. In accordance with Eq. 37, the very scale of the eddies militates against instantaneous dissipation in loco, giving the energy quanta invested into the vorticing units a certain period of life, before they are "ground up," in the process of swarming, and finally extinguished.

The nature of the phenomena involved in the deformation work during the conversion may be appraised quantitatively. Since, with Eq. 2, the energy loss in Eq. 4 equals  $\frac{\tau_o}{R} A U = \tau_o P_w U$  the energy available in the conversion zone, per unit of wall surface, may be expressed as

$$\frac{W_c}{P_w} = |\alpha \tau_o U| \dots \dots \dots (47)$$

in which  $\alpha = W_c/W_o$ , a fractional coefficient on the order of 0.8. The rate of deformation work per unit volume in the strained boundary vortex sheet (Fig. 18(a)), appraised on the basis of Eq. 12b, is

$$\mu \left( \frac{du}{dy} \right)_{\rightarrow 0}^2 = \mu \left( \frac{\tau_o}{\mu} \right)^2 = \frac{\tau_o^2}{\mu} \dots \dots \dots (48)$$

Since deformation work at such rate is called to absorb the energy amount indicated by Eq. 47, the volume of the deformable layer (Fig. 18) must measure transversely

$$y_d = \frac{\alpha \tau_o U}{\frac{\tau_o^2}{\mu}} = \frac{\alpha \mu U}{\tau_o} = \frac{\alpha U \nu}{u_f^2} = \frac{\nu}{u_f} \alpha \sqrt{\frac{2}{c_f}} \dots \dots \dots (49a)$$

Eq. 49a acquires a special significance when reduced to the terms of the so-called "friction distance" similitude parameter<sup>24</sup>

$$y_f = \frac{y u_f}{\nu} \dots \dots \dots (49b)$$

For smooth surfaces, flow is wholly shaped by the aforesaid similitude parameter. Indeed the thickness of the laminar sublayer  $\delta$  corresponds to a parametrical value of  $y_{fs} \approx 8$ , and the transverse latitude of the entire boundary zone, between the wall and the fully turbulent inner region, to  $y_{fbz} \approx 30$ . The in-between reach ( $8 < y_f < 30$ ) constitutes a "transitory" zone. The physical significance of this zone, which lies outward from the strictly laminar sublayer and inward from the turbulent reach where flow is fully "infested" with eddies, would be that of a turbulence generating region, within which eddies are formed and from which they are intermittently released.

In Eq. 49a, the thickness of the deformable viscous sheet, required for absorbing the convertible energy  $W_c$  in Eq. 47, corresponds to

$$y_{fd} = \frac{y_d u_f}{\nu} = \alpha \sqrt{\frac{2}{c_f}} \dots \dots \dots (50a)$$

For smooth pipes, with  $c_f$  between 0.0015 and 0.0025, and  $\alpha \approx 0.8$ ,

$$y_{fd} = 0.8 \sqrt{\frac{2 \cdot 10^4}{15 - 25}} \approx 29 - 23 \dots \dots \dots (50b)$$

commensurate with the thickness of the boundary zone. About a third of  $y_{fd}$  in Eq. 50b is assignable to the laminar sublayer, in which dissipation occurs in loco in accord with Eq. 12b. This leaves about two thirds, or slightly more, of  $W_c$  to be spent in the transitory zone for actual conversion into turbulence. Again a portion of this remaining energy must be dissipated locally in the course of rolling up the eddies, which means that only the smaller part of the flow energy  $W_c$ , first available in the conversion zone, finally reappears in turbulent eddying form.

*Transmittance Losses.*—In the central "turbulent" region the contributions of the viscous shear  $\mu \frac{d\bar{u}}{dy}$  in Eq. 45 are practically nil. Equally negligible are the energy losses  $\mu \left( \frac{d\bar{u}}{dy} \right)^2$ , computed from Eq. 12b by using for  $\frac{d\bar{u}}{dy}$  the observed local values of the main flow. It would be natural, on the other hand, to gage

the  $w$ -rates, as they actually appear in Figs. 17 and 11, in terms of the same "mixing" agencies, which shape the "turbulence" stress. For example, one may appraise the "mixing action" in the "Prandtl" manner,<sup>25</sup> assuming that a molar element  $m_1$  (Fig. 19(a)), initially belonging to filament (1) and traveling at a velocity  $u_1$ , is moved by a transverse impulse over a "mixing distance"

<sup>24</sup> "The Mechanics of Turbulent Flow," by B. A. Bakhmeteff, Princeton Univ. Press, Princeton, N. J., 1936, pp. 43 and 47.

<sup>25</sup> *Ibid.*, p. 40.

$l'$  into filament (2), where it becomes "embedded" acquiring the velocity  $u_2$  and suffering a velocity change:

$$|\Delta u| = |u_1 - u_2| = \left| l' \frac{d\bar{u}}{dy} \right| \dots \dots \dots (51)$$

Using the latter for  $u'$  in the expression for Reynolds "momentum transfer"—

$$\tau = \rho \overline{u'v'} \dots \dots \dots (52)$$

—assuming further that the transverse exchange intensity  $v'$  is of the same order and indeed is proportional to  $u'$ ; and finally absorbing the proportionality factor by a proper adjustment of the mixing scale, one ends with

$$\tau = \rho l' \frac{d\bar{u}}{dy} \times C l' \frac{d\bar{u}}{dy} = \rho l^2 \left( \frac{d\bar{u}}{dy} \right)^2 \dots \dots \dots (53)$$

in which  $l$  is the Prandtl "mixing length" and  $C$  is the proportionality factor.

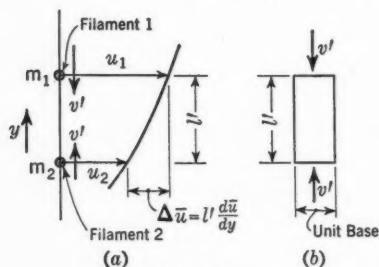


FIG. 19

Physically the change in the velocities due to the embedment is caused by viscous attrition that occurs during abrupt transverse displacements. Circumstances, in fact, are analogous to the case of impact with lasting constraints, as in the familiar instance of "inelastic bodies." Energy losses for the latter are expressed by the Carnot principle,<sup>22</sup> according to which the loss suffered by an element of mass  $m$ , the velocity of

which is altered from  $u_1$  to  $u_2$ , equals  $m(u_1 - u_2)^2/2$ .

In applying the Carnot principle to the block in Fig. 19(b), the aggregate mass exchanged per unit time across the upper and lower base is  $2 \rho |v'|$ , and the lost velocities equal  $\left| l' \frac{d\bar{u}}{dy} \right|$ , thus making the energy loss (by substituting Eq. 53):

$$2 \rho |v'| \frac{1}{2} \left( l' \frac{d\bar{u}}{dy} \right)^2 = \rho l^2 \left| \frac{d\bar{u}}{dy} \right|^3 l' = \tau \frac{d\bar{u}}{dy} l' \dots \dots \dots (54)$$

Dividing by the volume  $l' \times 1$ , one obtains the unit rate:

$$w \text{ (Carnot)} = \tau \text{ (turbulent)} \frac{d\bar{u}}{dy} = w_s \dots \dots \dots (55)$$

revealing that the "Carnot" loss, involved in the "mixing activity," is expressed by the same relation, which, in Eq. 34, determines the spending rate. In this light the quantity  $w_s$  (Eq. 34) in the central turbulent transmittance region, may be qualified as constituting a "mixing" or a "stress-forming" loss. Note, in particular, that the velocities altered through attrition in transverse embedment are those of the main flow. Physically, therefore, mixing action ap-

<sup>22</sup> "Traité de Mécanique Rationnelle," by Paul Émile Appell, Paris, France, 1911, Vol. II, p. 508.

pears to imply a direct dissipation of flow energy into molecular heat, arising from the transporting action of the large-size eddies. Such convective transfer of molar masses by the large vorticing units constitutes specifically the function of so-called "turbulent diffusion." Accordingly, the local loss of flow energy in the inner turbulent zone of a cross section, determined by Eq. 55, is a dissipation rate attendant upon "turbulent diffusion" and dependent in magnitude on the intensity of the latter.

### 13. APPLICATIONS TO ENGINEERING PRACTICE

*Broad Natural Watercourse.*—Consider a river sufficiently broad and regular so that the bottom traction force on a unit-area column (Fig. 20) can be expressed by the customary du Boys formula

$$\tau_o = -\gamma S y'_o \dots \dots \dots (56)$$

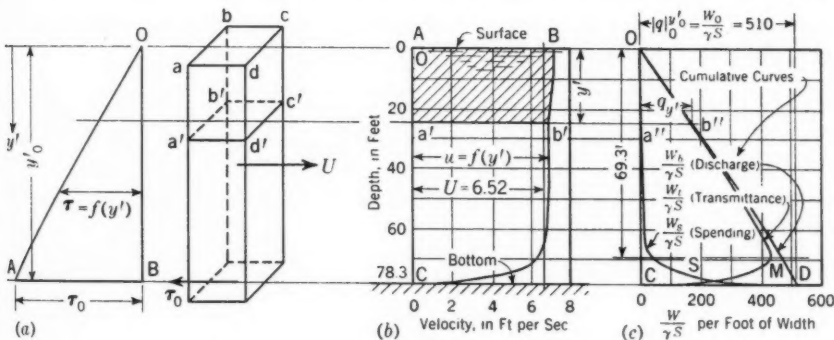


FIG. 20

which, with linear stress distribution, makes the unit shearing force in a plane  $a'b'c'd'$  equal to

$$\tau = \tau_o \frac{y'}{y'_o} = \gamma S y' \dots \dots \dots (57)$$

In other words, the stress diagram  $\tau$  is the product of the multiplier  $\gamma S$  by the depth. The observed velocity profile  $u = f(y')$  in Fig. 20(b) serves for plotting the cumulative discharge curve OD in Fig. 20(c), the ordinates of which ( $a''b''$ ) for  $y'$  represent the volume of flow:

$$|q|_0^{y'} = \int_0^{y'} u dy' = y' U_{av} \dots \dots \dots (58)$$

as indicated by the shaded area  $ABb'a'$  in Fig. 20(b).

In the light of Eqs. 7 and 9, whereas the total flow energy borrowed and spent in the unit prism of Fig. 20(a) is

$$W_o = \gamma S |q|_0^{y'_o} = \gamma S y'_o U \dots \dots \dots (59a)$$

the cumulative borrowed quantity—

$$|W_b|_0^{y'} = \gamma S |q|_0^{y'} \dots \dots \dots (59b)$$

—is directly represented by the cumulative discharge curve OD in Fig. 20(c) to the scale  $\frac{W_o}{\gamma S}$ . Likewise, with Eqs. 18 and 57, the energy rate cumulatively transmitted from the block  $y'$  deep across  $a'b'c'd'$  is

$$|W_t|_0^{y'} = -|\tau u|_0^{y'} = \gamma S y' u \dots \dots \dots (60)$$

represented (except for a common multiplier  $\gamma S$ ) in Fig. 20(c) by the curve OMC. The ordinates of this curve constitute the product of the local velocity by the respective depth, or the respective moment of the local velocity vector with regard to the surface of the streaming. The energy-spending curve  $W_s$  is obtained by subtracting, point by point,  $W_t$  from  $W_b$ , making

$$|W_s|_0^{y'} = \gamma S (|q|_0^{y'} - y' u) = \gamma S y' (U_{y'} - u) \dots \dots \dots (61)$$

In Eq. 61 the quantity in parentheses represents the difference between the partial mean velocity  $U_{y'}$  for the block (average ordinate of the shaded figure), and the local velocity for  $y'$  (ordinate  $a'b'$  in Fig. 20(b)).

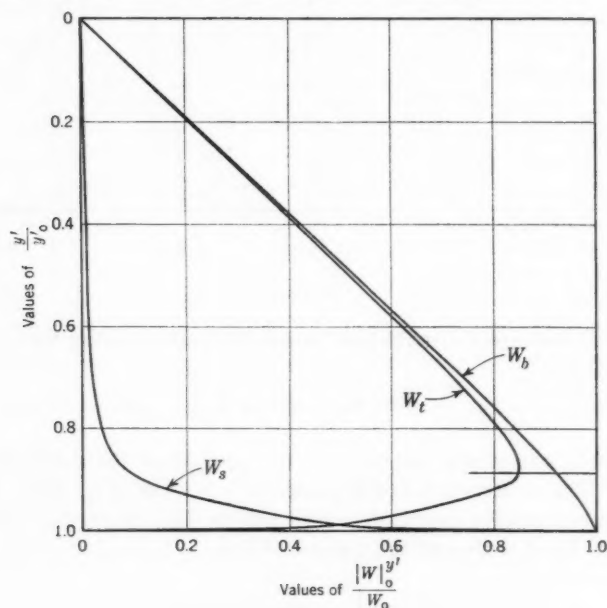


FIG. 21

The particular velocity profile, presented in Fig. 20(b), is for a large southern river during a moderately high stage. With a slope of 0.35 ft per mile, the multiplier in Eqs. 57 to 61 is  $\gamma S = 62.4 \times 0.66 \times 10^{-4} = 0.0041$ ; and its reciprocal is  $\frac{1}{\gamma S} \approx 244$ .

In Fig. 21, the curves of Fig. 20 are replotted in terms of dimensionless ratios. The features characteristic are typical and shed considerable light on

the eventual circumstances which condition the "working" of a watercourse. In fact, the reversal point is located at a depth of 69 ft to 70 ft, or at about 11% to 12% of the total depth (78.3 ft) from the bottom. Of the total energy  $W_o$  (Eq. 59a), nearly 92% is borrowed within the transmittance zone and of this about 90% is conveyed by transmittance for spending in the lower 8 ft to 9 ft of the flow. The aggregate transmittance loss is about  $8\frac{1}{2}\%$  of the total. In fact, the relatively small "slip" loss is suggested a priori by the shape of the velocity curve. In the upper regions this curve exhibits only slight deviations from a vertical, and actually begins to show a pronounced decline of the local velocity only in the lower reaches, below  $y' = 65$  ft. Actually over a depth of 70% of the total depth the cumulative loss is only 2% of  $W_o$ . It is less than 4% for 0.8  $y'_o$ , and 12% for 0.9  $y'_o$ . From there on, the spending increases rapidly. The most intensive exchange is in the closest proximity to the bottom, the spending in the last foot of the total  $y'_o = 78.3$  ft constituting 51% of  $W_o$ . An analogous procedure applies to circular conduits.

#### 14. GENERAL TREATMENT OF BOUNDARY LAYERS

It is assumed that the reader is familiar with the general concept and theory of flow in a boundary layer. A general account is given in several texts.<sup>27,28,29</sup>

To elucidate the process of energy loss in boundary layers, consider the two-dimensional motion next to a plate (see Fig. 22), under the customary

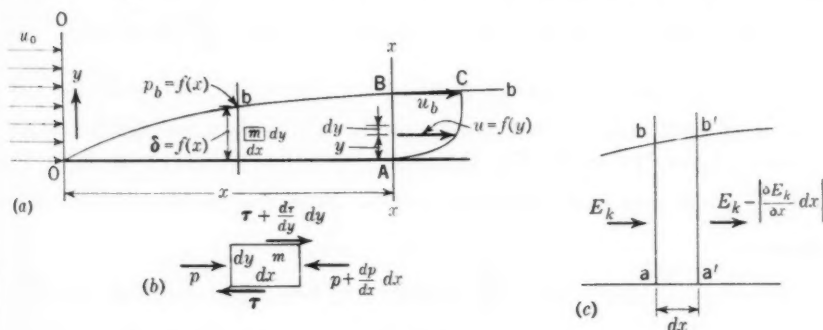


Fig. 22

premises that motion in the transverse sense is negligible, and that the pressures in a cross section do not differ from the pressure  $p_b = f(x)$  at the border point  $b$ . In Fig. 22, and elsewhere in this paper, the reasoning is applied to a layer of unit width in the direction perpendicular to the plane of the figure.

The dynamic equilibrium for an element (Fig. 22(b)) is expressed by equating the mass-acceleration quantity  $\rho \, dx \, dy \, \frac{du}{dt}$  to the resultant of the forces, of

<sup>27</sup> "Fundamentals of Hydro- and Aeromechanics," based on lectures of L. Prandtl, by O. G. Tietjens, translated by L. Rosenhead, 1st Ed., McGraw-Hill Book Co., Inc., New York and London, 1934, p. 58.

<sup>28</sup> "The Mechanics of Viscous Fluids," Div. G in "Aerodynamic Theory," by L. Prandtl, W. F. Durand, editor in chief, Springer, Berlin, Germany, 1935, Vol. III, p. 80.

<sup>29</sup> "Fluid Mechanics for Hydraulic Engineers," by Hunter Rouse, 1st Ed., McGraw-Hill Book Co., Inc., New York and London, 1938, p. 192 et seq.

which  $-\frac{dp_b}{dx} dx dy$  is the pressure component, and  $\frac{d\tau}{dy} dy dx$  is the friction resultant. Reduced to a unit volume of fluid, the equation is

$$\rho \frac{du}{dt} = -\frac{dp_b}{dx} + \frac{d\tau}{dy} \dots \dots \dots (62)$$

For the usual case of  $u_b = \text{constant}$ , and  $\frac{dp_b}{dx} = 0$ , Eq. 62 becomes

$$\rho \frac{du}{dt} = \frac{d\tau}{dy}; \quad \frac{d\tau}{dy} + \left( -\rho \frac{du}{dt} \right) = 0 \dots \dots \dots (63)$$

indicating that, in the absence of a pressure gradient, motion in the boundary layer is sustained by the inertia of the oncoming flow. The negative friction resultant  $\frac{d\tau}{dy}$  (see Eqs. 10) is offset by an appropriate retarding of the motion,

with the positive "inertial force"  $\left( -\rho \frac{du}{dt} \right)$  balancing the negative friction traction. Obviously the work implied in overcoming frictional resistances is performed at the expense of the kinetic energy of the flow, the initial store of which is gradually reduced. With regard to an elementary cross-sectional block (Fig. 22(c)), the kinetic energy  $E_k$ , entering per unit time across  $ab$ , exceeds the energy leaving the block across line  $a'b'$  by  $\left| \frac{dE_k}{dx} \right| dx$ , in which the

quotient  $\left| \frac{dE_k}{dx} \right|$  measures the energy withdrawn (borrowed) and spent per unit time in a boundary layer block of unit length. It has the same significance for boundary layer flow as  $W_o$  (Eq. 4) has for established uniform motion, except that  $\left| \frac{dE_k}{dx} \right|$  will vary from section to section. Since the quotient is essentially negative,

$$-\frac{dE_k}{dx} = f(x) = (W_\delta)_x \dots \dots \dots (64)$$

the inner subscript  $\delta$  suggesting that  $W_\delta$  covers the entire transverse extent of the layer, whereas the outer subscript indicates specifically the location of the particular unit block under consideration. On the other hand,  $|W_\delta|_0^x$  will designate the aggregate energy borrowed and spent in a unit of time in the entire body of the boundary layer OBA between O and  $x$ . Obviously for  $\frac{dp}{dx} = 0$ ,  $|W_\delta|_0^x$  is the difference between the kinetic energy  $\rho \int_0^\infty u \frac{u^2}{2} dy$  flowing across AB equal to the energy  $\rho \int_0^\infty u \frac{u_o^2}{2} dy$ , originally carried by the same volume of flow in the on-coming undisturbed state, across O-O. In other words,

$$|W_\delta|_0^x = \rho \int_0^\infty u \left( \frac{u_o^2 - u^2}{2} \right) dy = \frac{\rho u_o^3}{2} \int_0^\infty \frac{u}{u_o} \left[ 1 - \left( \frac{u}{u_o} \right)^2 \right] dy \dots (65)$$

The quantity  $E_k$  includes energy carried across the cross section from the plate to infinity. Inasmuch as there is no appreciable change of velocity outside of the boundary layer, one may substitute for the right side of Eq. 65:

(62) 
$$\frac{\rho u_o^3}{2} \int_0^y \frac{u}{u_o} \left[ 1 - \left( \frac{u}{u_o} \right)^2 \right] dy.$$
 Eq. 65 complements the well-known von Kármán expression

$$F_x = \rho u_o^2 \int_0^x \frac{u}{u_o} \left( 1 - \frac{u}{u_o} \right) dy \dots \dots \dots (66)$$

(63) which serves to determine the skin friction resultant over the length  $x$  in terms of the momentum lost by the flow between 0 and  $x$ . Note that Eqs. 65 and 66 are quite general and therefore are applicable to boundary layers of all types

whether laminar or turbulent, subject only to the limitation  $\frac{dp_b}{dx} = 0$ . Unfortunately,

present knowledge of turbulent boundary layers is anything but complete, and an adequate analysis of the energy exchanges is not possible. On the other hand, for the case of a laminar layer next to a flat plate with

$\frac{dp_b}{dx} = 0$ , detailed theoretical calculations are available, and are in excellent accord with repeated observations. For example, theoretical and experimental aspects have been presented in detail by Hugh L. Dryden,<sup>30</sup> whose report, as well as subsequent papers by the same author,<sup>16</sup> contain the more recent interpretations of boundary layer motion, particularly with regard to "transition."

#### 15. ENERGY BALANCE IN LAMINAR BOUNDARY LAYERS WITH $\frac{dp}{dx} = 0$

Basically, the flow pattern is conditioned by the parameter

$$\sqrt{\frac{x \nu}{u_o}} = x \sqrt{\frac{\nu}{x u_o}} = \frac{x}{\sqrt{R_x}} \dots \dots \dots (67)$$

(64) in which  $R_x = \frac{x u_o}{\nu}$  is the so-called "distance Reynolds number" of the boundary layer. In particular, the transverse velocity profile  $u = f(y)$  at a station  $x$  (Fig. 22(a)) is determined by the dimensionless quantity

$$\eta = \frac{y}{\sqrt{\frac{x \nu}{u_o}}} = \frac{y}{x} \sqrt{R_x} \dots \dots \dots (68)$$

so that the ratio  $\phi$  of a local velocity  $u$  distant  $y$  from the plate, to the constant (border) velocity  $u_o$  ( $\phi = \frac{u}{u_o}$ ) is solely a function of  $\eta$ . In any two cross sections, if the ordinates were chosen to make  $\eta_1 = \left( \frac{y_1}{x_1} \right) \sqrt{R_{x_1}}$  and

<sup>30</sup> "Air Flow in the Boundary Layer Near a Plate," by Hugh L. Dryden, *Technical Report No. 562*, National Advisory Committee for Aeronautics, U. S. Government Printing Office, Washington, D. C., 1937, p. 339.

$\eta_2 = \left(\frac{y_2}{x_2}\right) \sqrt{R_{x_2}}$  identical, the respective velocity ratios  $\phi_1 = \frac{u_1}{u_o}$  and  $\phi_2 = \frac{u_2}{u_o}$  would be the same. Velocity profiles with such similarity of shape are said to be "affine," permitting the representation of the  $u = f(y)$ -curve for any cross

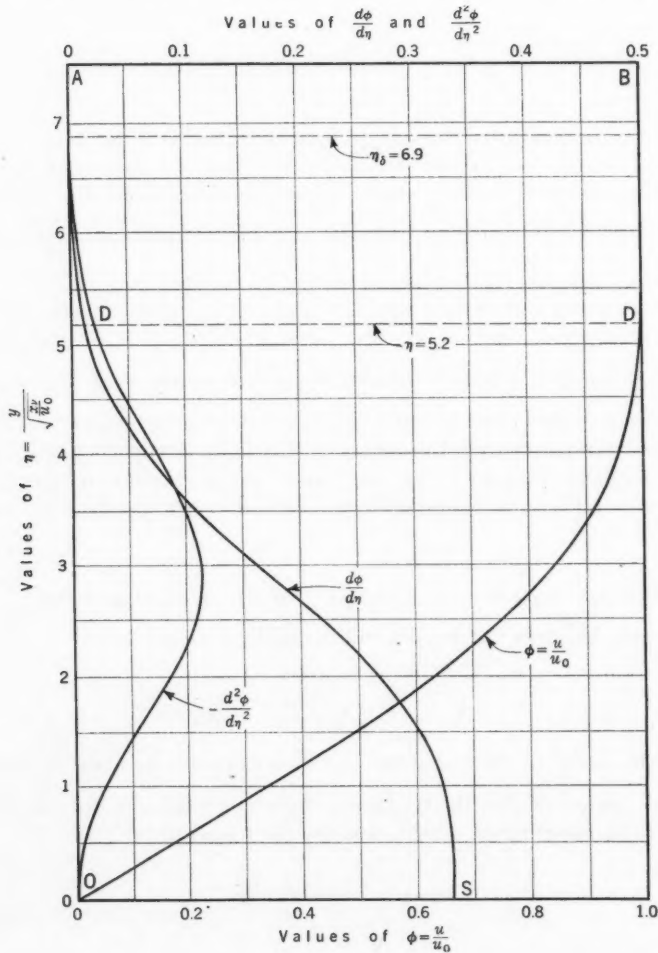


FIG. 23

section by the single generalized outline plotted as  $\phi\left(\frac{y}{\sqrt{\frac{\nu x}{u_o}}}\right) = f(\eta)$  in Fig. 23. Affinity obviously applies also to the derivative quantities

$$\frac{d\phi}{d\eta} = \frac{du}{dy} \sqrt{\frac{\nu x}{u_o^3}}; \quad \frac{d^2\phi}{d\eta^2} = \frac{d^2u}{dy^2} \frac{\nu x}{u_o^2} \dots \dots \dots (69)$$

the generalized outlines for which are also presented in Fig. 23 (based on a tabulation of the original Blasius data<sup>31</sup> included by the Dryden report.<sup>30</sup> The values of  $\frac{d^2u}{dy^2}$  throughout the layer are negative. In Fig. 23 the positive of the curve corresponds to  $-\frac{d^2\phi}{d\eta^2}$ .) In particular the dotted line D-D in Fig. 23 is drawn at a transversal distance  $\eta = 5.2$ , suggested by Prandtl as a conventional measure for the boundary layer thickness  $\delta = 5.2 \sqrt{\frac{x\nu}{u_o}}$ . At such distance, the local value of  $u$  differs from the border quantity  $u_o$  by the negligible amount of 0.5%. Fig. 23 discloses, however, that the derivative values at  $\eta = 5.2$  are still quite substantial, and may weigh accordingly in the eventual energy balances. A more appropriate conventional value for the laminar boundary layer thickness would be  $\eta_\delta = 6.9$  making  $\delta = 6.9 \sqrt{\frac{x\nu}{u_o}}$ . The particular value of  $\eta_\delta = 6.9$  is chosen, so as to give  $\delta$  four times the so-called "displacement thickness"  $\delta_d = 1.73 \sqrt{\frac{\nu x}{u_o}}$ . The Prandtl  $\eta_\delta = 5.2$  corresponds to  $\delta = 3 \delta_d$ .

For "affine" profiles, the energy equations take definite numerical form. In fact, substituting  $\phi u_o$  for  $u$ , and  $\eta \sqrt{\frac{x\nu}{u_o}}$  for  $y$ , the aggregate energy loss (Eq. 65) becomes

$$|W_\delta|_0^x = \frac{\rho u_o^3}{2} \sqrt{\frac{x\nu}{u_o}} \int_0^\infty \phi (1 - \phi^2) d\eta = \frac{\rho x u_o^3}{2 \sqrt{R_x}} \omega = 0.522 \frac{\rho x u_o^3}{\sqrt{R_x}} \quad (70)$$

in which  $\omega$  is a dimensionless factor representing the value of the integral

$$\omega = \int_0^\infty \phi (1 - \phi^2) d\eta \dots \dots \dots (71)$$

Eq. 71 is dependent solely on the shape of the "affine" profile. For the laminar outline of Fig. 23, the quantity  $\omega$  equals 1.044.

To obtain the local cross-sectional loss (Eq. 64), one proceeds by differentiating Eq. 70, which makes

$$(W_\delta)_x = \frac{d|W_\delta|_0^x}{dx} = \frac{\omega \rho u_o^3}{2} \frac{d}{dx} \sqrt{\frac{x\nu}{u_o}} = 0.261 \frac{\rho u_o^3}{\sqrt{R_x}} \dots \dots \dots (72)$$

In Eqs. 70 and 72—

$$|W_\delta|_0^x = 2 x (W_\delta)_x \dots \dots \dots (73)$$

For establishing the interrelation between the energy loss and the skin friction resultant, note that, numerically, the von Kármán expression (Eq. 66)

<sup>31</sup> "Air Flow in the Boundary Layer Near a Plate," by Hugh L. Dryden, *Technical Report No. 562*, National Advisory Committee for Aeronautics, U. S. Government Printing Office, Washington, D. C., 1937, p. 361.

equals

$$F_x = 0.664 \frac{x}{\sqrt{R_x}} \rho u_o^2 \dots \dots \dots (74)$$

and the local wall stress (friction force per unit length of plate at the distance  $x$ ) is

$$\tau_o = \frac{dF_x}{dx} = 0.332 \frac{\rho u_o^2}{\sqrt{R_x}} = \frac{F_x}{2x} \dots \dots \dots (75)$$

Substituting Eq. 75 into Eqs. 70 and 72, one finds that

$$|W_\delta|_0^x = \frac{\omega u_o \rho u_o^2 x}{2 \sqrt{R_x}} \approx \frac{3}{4} \omega F_x u_o = 0.786 F_x u_o \dots \dots \dots (76a)$$

and

$$(W_\delta)_x = \frac{\omega u_o \rho u_o^2}{4 \sqrt{R_x}} \approx \frac{3}{4} \omega \tau_o u_o = 0.786 \tau_o u_o \dots \dots \dots (76b)$$

## 16. THE LOCAL UNIT RATES

Local borrowing in a boundary layer is expressed by the same formula (Eq. 11) as in uniform established motion, although, with  $\frac{dp_b}{dx} = 0$ , the work absorbed by the friction tractions is supplied wholly from the kinetic energy store.

In particular, for a laminar layer, using Eq. 69 in Eq. 11, one obtains

$$w_b = -\mu \frac{d^2 u}{dy^2} u = \frac{\rho u_o^3}{x} \left( -\phi \frac{d^2 \phi}{d\eta^2} \right) \dots \dots \dots (77)$$

the multiplier of the parentheses group constituting (in view of Eq. 72):

$$\frac{\rho u_o^3}{x} = \frac{4}{\omega} \sqrt{\frac{u_o}{\nu x}} (W_\delta)_x = 3.83 \frac{\sqrt{R_x}}{x} (W_\delta)_x \dots \dots \dots (78)$$

Furthermore, dividing  $(W_\delta)_x$  in Eq. 72 by the transverse dimension of the layer  $\delta = 6.9 \sqrt{\frac{x \nu}{u_o}}$  (see paragraph containing Eq. 69), one obtains an average unit rate of

$$(w_\delta)_x = \frac{(W_\delta)_x}{\delta} = \frac{\omega}{27.6} \frac{\rho u_o^3}{x} = \frac{1}{26.4} \frac{\rho u_o^3}{x} \dots \dots \dots (79)$$

which is a quantity analogous to  $w_o$  (Eq. 8) for uniform flow. Substituting Eq. 79 into Eq. 77:

$$w_b = (w_\delta)_x \frac{27.6}{\omega} \phi \left( -\frac{d^2 \phi}{d\eta^2} \right) = 26.4 (w_\delta)_x \epsilon'_b \dots \dots \dots (80)$$

in which

$$\epsilon'_b = \phi \left( -\frac{d^2 \phi}{d\eta^2} \right) \dots \dots \dots (81)$$

is a dimensionless function, represented in Fig. 24 by the curve AaO. This curve is obtained by multiplying the ordinates of the curves Ao and Bo in Fig. 23. Obviously,  $\epsilon'_b$ , in terms of  $26.4 (w_b)_x$ , denotes the distribution of the borrowing process in all laminar boundary layers of the type  $\frac{dp_b}{dx} = 0$ . The

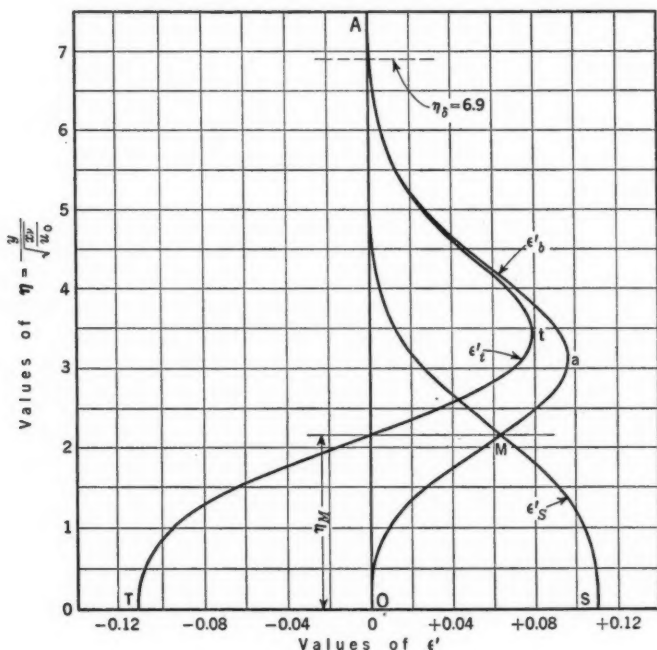


FIG. 24

local borrowing is zero near the wall, and it also declines into insignificance beyond  $\eta_b = 6.9$ . The maximum intensity of the borrowing function is reached in the intermediate region near  $\eta = 3 - 3.25$ .

*The Spending Rate.*—In a laminar layer, with the local spending measured by the work involved in viscous deformation (Eq. 12b), one obtains, with Eq. 69,

$$w_s = \mu \left( \frac{du}{dy} \right)^2 = \frac{\mu u_o^3}{\nu x} \left( \frac{d\phi}{d\eta} \right)^2 = \frac{\rho u_o^3}{x} \left( \frac{d\phi}{d\eta} \right)^2 = 26.4 (w_b)_x \epsilon'_s \dots (82)$$

Again, the dimensionless function

$$\epsilon'_s = \left( \frac{d\phi}{d\eta} \right)^2 \dots \dots \dots (83)$$

represented in Fig. 24 as the curve AS, describes the spending process in generalized universal terms. Similar to uniform motion, the spending of energy

is concentrated near the solid boundary. With  $\eta_M = y_M \sqrt{\frac{x}{u_o}}$ , the intersection point M, corresponding to  $w_b = w_s$ , indicates the region within which spending exceeds the energy amounts obtained by local borrowing.

*The Transmittance Rate.*—By subtracting curve AS (Fig. 24) from curve AaO, one obtains the outline curve AtT, or  $\epsilon'_t$  which defines the local transmittance function. Expressed algebraically,

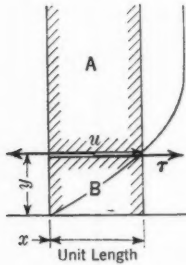


FIG. 25

$$w_t = w_b - w_s = \frac{\rho u_o^3}{x} \left[ -\phi \frac{d^2\phi}{d\eta^2} - \left( \frac{d\phi}{d\eta} \right)^2 \right]$$

$$= \frac{\rho u_o^3}{x} \frac{d}{d\eta} \left( -\phi \frac{d\phi}{d\eta} \right) = 26.4 (w_b)_x \epsilon'_t \dots (84)$$

*Cumulative Energy Rates.*—In presenting the cumulative energy quantities, it is expedient to add in the direction from the unaffected outward zone toward the plate, thus making the resulting outlines for a boundary layer directly comparable to those of Figs. 1 and 2. In fact, using Eq. 69 with Fig. 25 one obtains for the transmittance rate (compare Eq. 18):

$$|W_t|_y^\eta = \left| -u \tau \right|_y^\eta = \left| \phi u_o \mu \frac{du}{dy} \right|_y^\eta = \frac{\rho u_o^3}{x} \sqrt{\frac{x}{u_o}} \left| \frac{d\phi}{d\eta} \phi \right|_y^\eta$$

$$= \frac{4}{\omega} (W_b)_x \left| \phi \frac{d\phi}{d\eta} \right|_y^\eta = 3.83 (W_b)_x E'_t \dots (85)$$

in which  $E'_t$  is a dimensionless multiplier:

$$E'_t = \left| \phi \frac{d\phi}{d\eta} \right|_y^\eta \dots (86)$$

obtained from the data in Fig. 23, and traced as curve ATO in Fig. 26. Physically the quantity  $|W_t|_y^\eta$  indicates the work performed on the block B, Fig. 25, by the overlying fluid prism A. The other cumulative quantities take the form:

$$|W_b|_y^\eta = \int_0^\eta w_b dy = \frac{\rho u_o^3}{x} \sqrt{\frac{x}{u_o}} \int_0^\eta \left( -\phi \frac{d^2\phi}{d\eta^2} \right) d\eta$$

$$= \frac{4}{\omega} (W_b)_x \int_0^\eta \epsilon'_b d\eta = 3.83 (W_b)_x E'_b \dots (87a)$$

and

$$|W_s|_y^\eta = \int_0^\eta w_s dy = \frac{\rho u_o^3}{x} \sqrt{\frac{x}{u_o}} \int_0^\eta \left( \frac{d\phi}{d\eta} \right)^2 d\eta$$

$$= \frac{4}{\omega} (W_b)_x \int_0^\eta \epsilon'_s d\eta = 3.83 (W_b)_x E'_s \dots (87b)$$

in which the generalized functions  $E'_b$  and  $E'_s$  are numerically equal to the areas under the curves AO and AS in Fig. 24, traced in Fig. 26 as AbB and AsB.

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*The Energy Balance Equation.*—By analogy to Eq. 17, the local energy balance takes the generalized form

$$26.4 (w_b)_z \left\{ \epsilon'_b = \epsilon'_s + \epsilon'_t \right\} \dots \dots \dots (88)$$

that is,

$$\frac{\rho u_o^3}{x} \left\{ -\phi \frac{d^2\phi}{d\eta^2} = \left( \frac{d\phi}{d\eta} \right)^2 - \frac{d}{d\eta} \left( \phi \frac{d\phi}{d\eta} \right) \right\} \dots \dots \dots (89)$$

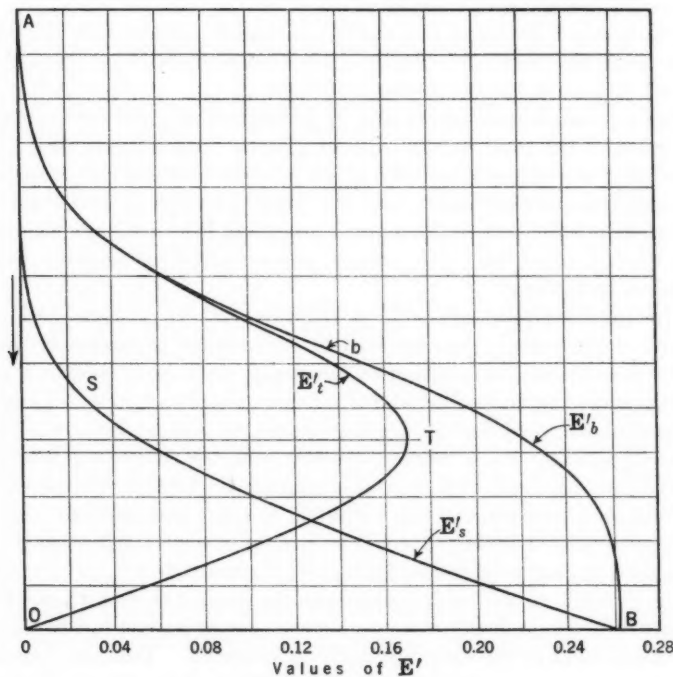


FIG. 26

The analogous symbolic formula expressing the relation between the cumulative energy quantities is:

$$3.83 (W_b)_z \left\{ E'_b = E'_s + E'_t \right\} \dots \dots \dots (90)$$

#### SUMMARY

1. In the mechanism of energy loss attendant to fluid friction, the "borrowing," or withdrawing of energy from the flow, and its spending do not coincide spacially. Spending is concentrated in the regions of high vorticity near the solid boundaries, whereas the borrowing takes effect principally in the central reaches of high local velocity.

2. The aforesaid implies an in-between "transmittance" function, by which surplus borrowed energy is transmitted to the wall region, where local borrowing is insufficient to cover the requirements for spending.

3. The aspects of the three consequent phases are determined by the internal stress structure and the concomitant velocity profile. The local energy balance equation

$$-u \frac{d\tau}{dy} = \tau \frac{du}{dy} + \frac{d}{dy} (-\tau u) \dots \dots \dots (91)$$

$(w_b) \quad (w_s) \quad (w_t)$

applies equally to laminar and turbulent patterns, and serves as the basis for the reasoning presented in the paper.

4. For laminar patterns the energy exchange process can be presented wholly in analytical form. In turbulent patterns the application of the equations must rely on detailed experimental data.

5. The outstanding characteristic of all turbulent patterns is that only a small portion of the total energy is spent directly within the central turbulent zone, and that the larger part (80% to 85%) is concentrated by transmittance for spending in a narrow reach near the solid boundary. This "conversion" zone is ostensibly the seat of the pivotal process in the turbulent mechanism—namely, that of converting flow energy, concentrated by transmittance, into turbulent eddying form.

6. The function of conversion in itself reverts to viscous action. The larger part of the energy, concentrated for conversion, is ostensibly dissipated in loco in the viscous deformation of the laminar sub-layer and in the course of forming eddies in the adjacent turbulence generating zone. Thus only a smaller part of the spent flow energy reappears after conversion in the turbulent form which animates the familiar turbulent manifestations and activates the convective mass transfer instrumental in shaping the turbulent stress structure.

7. The final dissipation of the turbulent energy, continuously replenished from the generating boundary zone, is by way of viscous attrition in the course of the swarming motion of the eddies across the main flow.

8. The direct local loss of flow energy in the central transmittance zone can be accounted for dynamically by applying the Carnot impact law to the process of mass embedment in the course of transverse mixing. This loss, attributable to the convective action of the larger eddies which shape the stress structure, is concomitant to turbulent diffusion.

9. In relation to natural watercourses, the transmittance of energy and its spending in the boundary zone, shed possible light on the "working" of a river in regard to bed-load movement.

10. The analysis of boundary layer motion is extended to consider the energy aspects of the flow patterns. In the case of a laminar layer with  $dp/dx = 0$ , a complete account of the energy exchange mechanism is revealed in generalized dimensionless form.

*Acknowledgments.*—The subject matter covered by the paper forms part of a broader investigation on the "Nature of Hydraulic Resistances" conducted under the general auspices of the Society's Committee on Hydraulic Research. For the velocity profile used in Fig. 20 the writers are gratefully indebted to the Mississippi River Commission of Vicksburg, Miss.

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# AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

## PAPERS

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### A PLAN FOR A MOVABLE DAM

BY ISAAC DEYOUNG,<sup>1</sup> M. AM. SOC. C. E.

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#### SYNOPSIS

There are two general conditions under which movable dams are operated: One in which it is required to close a channel in an orderly predetermined plan and without immediate urgency; and the other, in which a sudden release of the water in a canal caused by an accident due to the wrecking of a lock gate by collision or otherwise, requires a structure that can be operated in the rushing waters, checking the flow.

The plan for a movable dam presented herewith is of the latter type. It is an emergency dam, and is designed so that it can be put in place during the rush of waters thus stopping the uncontrolled stream. It consists of plate girders (which form the dam), connected by hinges fastened together at their flanges and folded together in small compass in the bottom of the lock. The dam is held in place by rigid anchors, hinged to the girders, and embedded anchorages under the floor of the lock. The dam is raised through multiple reaved cables by two hoisting engines. The raising operation required only a short time. A single operator in a central station manipulates the controls in an orderly way as soon as the upper canal is free from all floating craft.

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#### INTRODUCTION

The movable dam proposed in this paper is a development from the studies made in the early 1900's in connection with the design and construction of two movable dams for St. Marys Falls Canal, Michigan, in which the writer had a part. The writer's ideas leading to the proposed dam have been developed through the years intervening since that first experience.

#### FORMS OF MOVABLE DAMS

There are four general types of movable dams: (1) Overhead bridges built in conjunction with wickets, needles, and other devices; (2) submerged structures;

NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by July 1, 1945.

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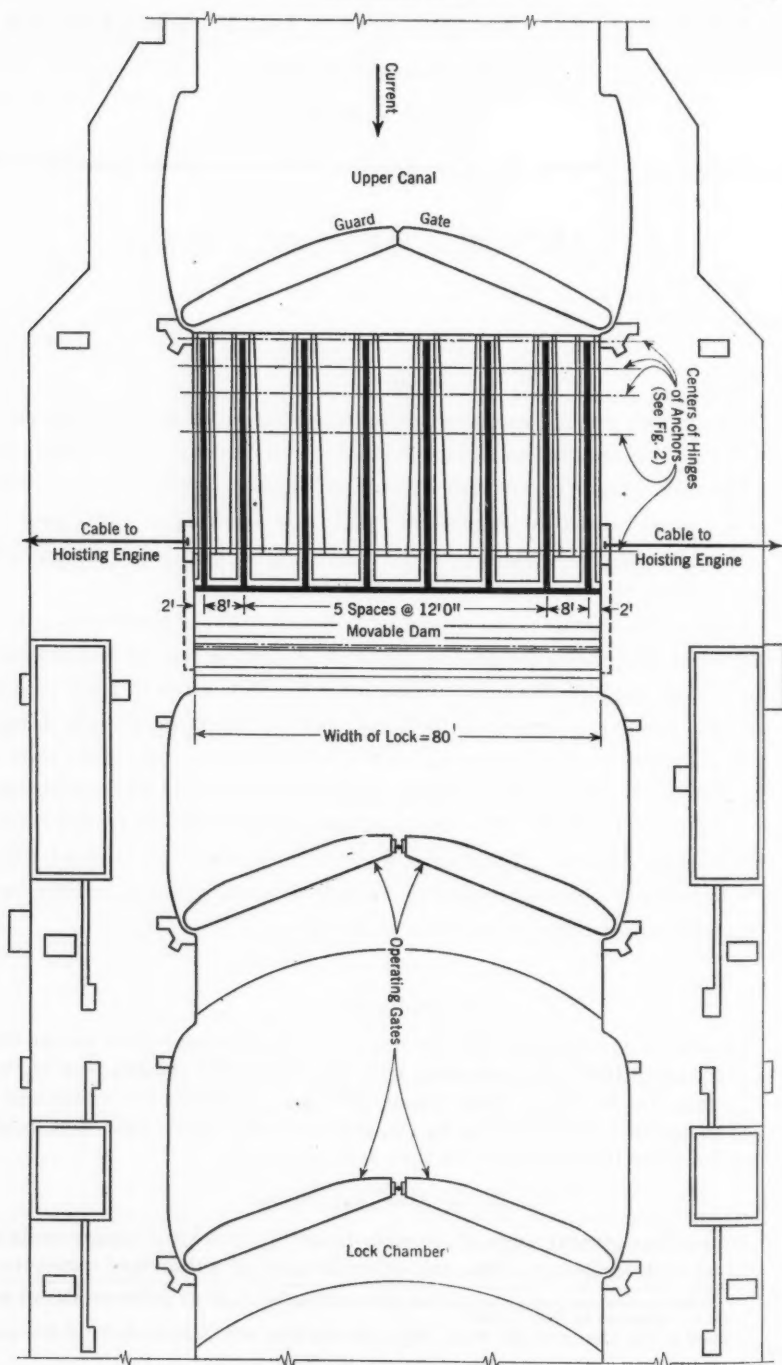


FIG. 1.—PLAN OF HYPOTHETICAL, FLEXIBLE SHUTTER DAM, DESIGNED FOR THE UPSTREAM END OF LOCK ST. MARYS FALLS, MICHIGAN

(3) floats; (4) lock gates (sheathed or skeleton), horizontal sliding gates, rolling segmental gates, and lift gates.

(1) Overhead bridges of the swing type, from which either horizontal or vertical stop planks, wickets, or needles are manipulated, are fairly sure to be operated so that the flow of water is checked gradually, thus minimizing the shock on all parts of the structure. The operating parts are readily accessible and easily repaired. Some of the disadvantages are that they are limited as to length of span, possibility that a sunken vessel in the locality may prevent the bridge from closing, and the lodgment of stone or other material on the sill.

(2) Among dams of the submerged type may be mentioned trestle dams, bear traps, Taintor gates, etc. The flexible shutter type of dam described in this paper belongs in this class (see Figs. 1 and 2). Some inherent objections to the typical submerged dam are that a dragging anchor may injure the structure and that, when damaged, the structure is relatively inaccessible for repairs. These objections, as applied to the flexible shutter dam, are discussed subsequently.

(3) Under the subject of floats, there is the barrier created by sinking a vessel of any shape. A circular caisson built of timber or steel can be used and floated into place at the jaws of the narrowest part of the canal or lock.

(4) Having two sets of operating gates at each end of the lock offers some safety because of the double barrier in front of a moving vessel. A gate located in the canal above the lock would accomplish a similar result, the gate being operated in such a manner as to be always closed during a danger period when the lock gates are open; but the operations under such conditions would be subject to excessive delays to navigation.

#### DESCRIPTION OF THE FLEXIBLE SHUTTER

The structure described in this paper is composed mostly of plate girders, fastened together on the downstream side by plate hinges extending the entire length of the girders (Figs. 2(a), 3, 4, 5, and 6). On the upstream side the dam is held by anchors connected to embedded anchorages. It is raised by two hoisting engines, through six-part cable, one on each side of the lock. In the lowered position, the dam is folded in small compass below the lock floor level (see Figs. 3 and 7). It is designed for a lock 80 ft wide, having 30 ft of water on the sills.

The selection of a type of movable dam should be based on the following requirements: (a) Certainty of operation when required; (b) reasonable facility of operation; (c) no obstruction to navigation; (d) ability to withstand shock; (e) simplicity of construction; (f) accessibility for inspection and repairs; (g) appearance; (h) freedom from enemy attack during stress of war; and (i) cost of construction and maintenance.

(a) A movable dam should be free from the possibility of being clogged or broken by the ordinary use of the canal or by the exigencies arising at the time of the accident.

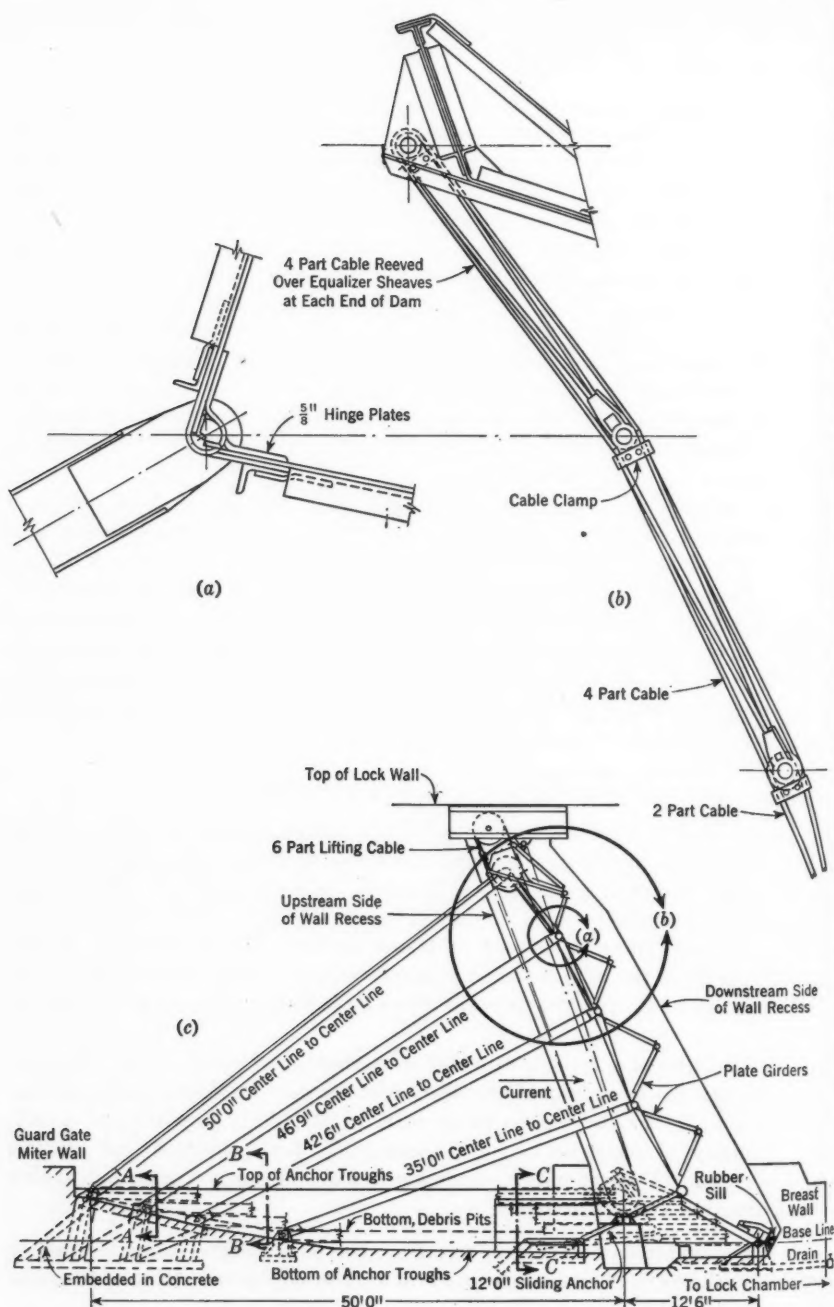


FIG. 2.—LONGITUDINAL SECTION OF DAM, SHOWING GENERAL OUTLINES AND LOCATION OF SECTIONS (FOR SECTIONS, SEE FIG. 7)

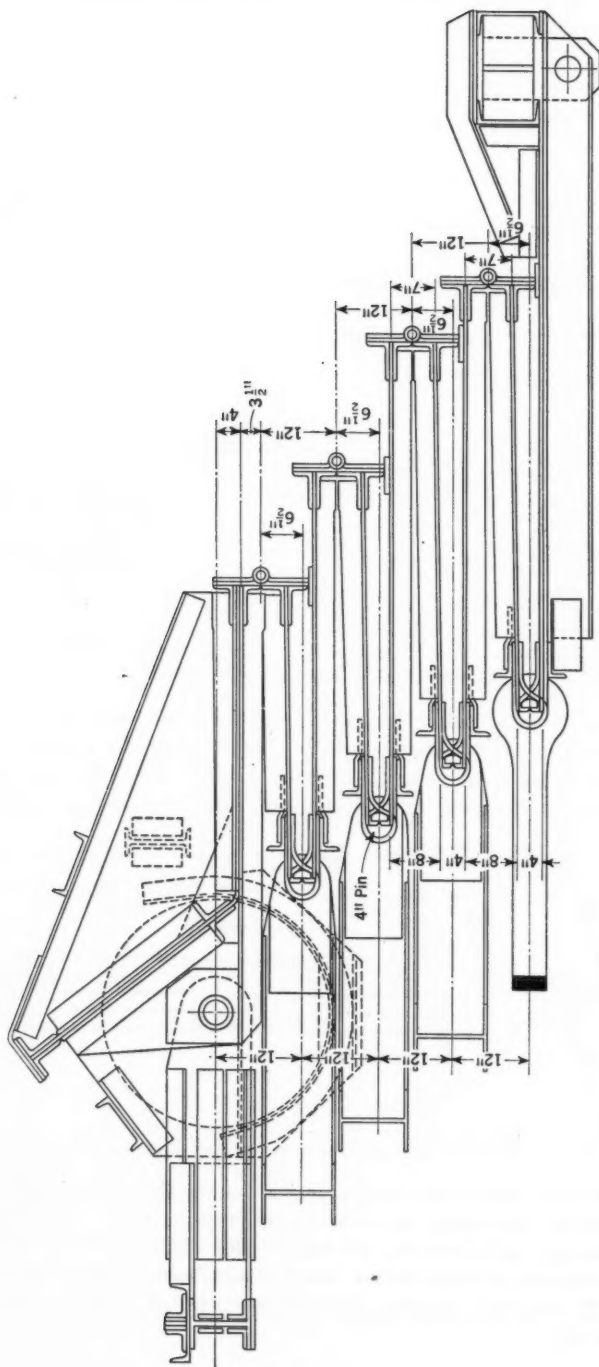


FIG. 3.—DETAILED VIEWS OF DAM IN A LOWERED POSITION

In the proposed plan the dam is folded in small compass in the bottom of the lock and protected by transverse breast walls, preventing damage from dragging equipment, such as anchors.

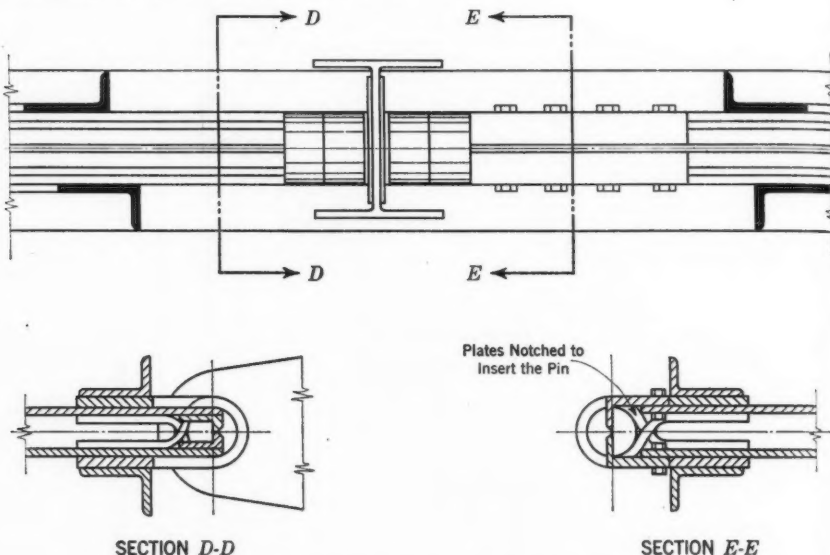


FIG. 4.—SECTIONS THROUGH UPSTREAM HINGES

As debris may be moved by the current created in the canal when a lock gate is wrecked, some provision should be made to prevent the collection of such debris on parts of the dam, so as to interfere with its successful operation. In the plan, the anchor slots, eight in number, are covered by the top anchor having

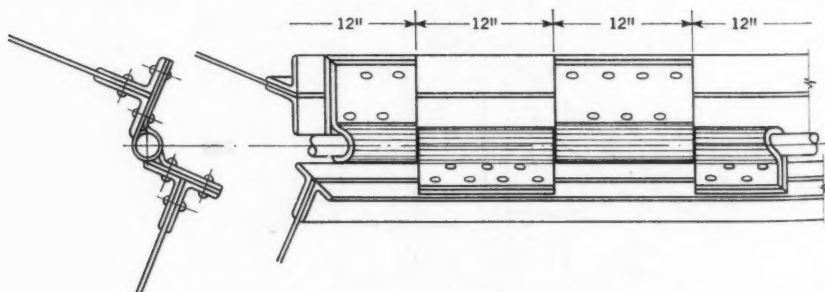


FIG. 5.—DOWNSTREAM HINGES, CONTINUOUS END TO END OF DAM

angle iron attached, whose outstanding legs cover the slots completely from end to end of anchors, preventing the debris from entering these slots (see Fig. 6). Also, funnel-shaped pits between the anchor slot walls catch this debris.

It is not considered likely that a vessel would become stuck, because then it would itself become a barrier, building up pressure sufficient to move it through the lock.

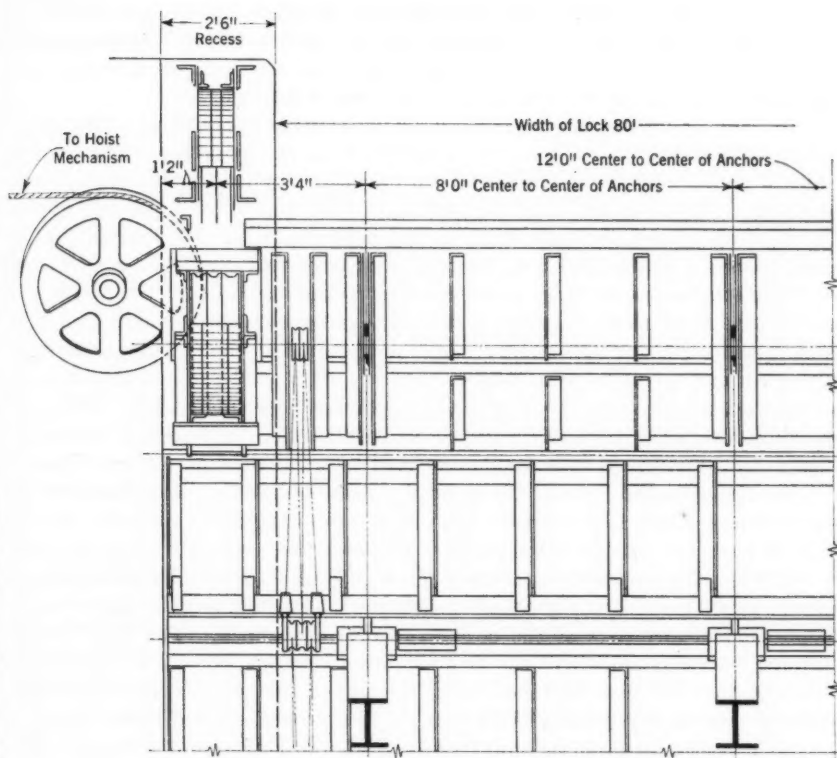


FIG. 6.—REAR ELEVATION, FACING DOWNSTREAM

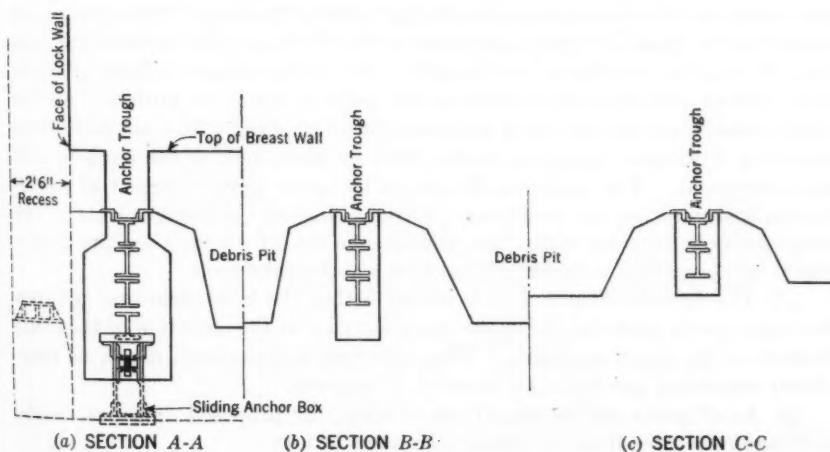


FIG. 7.—SECTIONS A-A, B-B, AND C-C, FIG. 2, SHOWING ANCHORS IN THE LOWERED POSITION

It would be desirable if the dam could be closed before the current in the canal has reached a dangerous velocity, but this ideal is impossible in any form of dam. The flexible shutter dam can be raised in a few minutes, and as soon as all floating craft in the upper canal has passed through.

(b) The operating mechanism should be simple, not likely to get out of order, and it should be easily accessible for quick operation. The flexible shutter dam is operated by means of two hoisting engines, one on each side of the lock. The operation may be performed from one station through appropriate electrical connections similar to the Selsyn method of operation. (The Selsyn system of signaling and control consists of two small specially wound induction motors electrically interconnected in such a way that the movement of one armature causes the other armature to move by a corresponding amount and in the same direction. One motor is operated at the sending point as a generator and is called the transmitter and the other is operated as a motor and is known as the receiver.) The dam is operated as a single unit. The plan provides for a dam in a lock 30 ft deep and 80 ft wide. In a lock of less depth and width, the operation may be performed by hand through a train of gears.

(c) The structure should be of such a form as to offer no obstruction to navigation. These requirements have been met in the flexible shutter dam in that all parts are outside of the limits of the navigable cross section of the lock.

(d) Since the proposed structure is held rigidly in place by substantial anchor beams, which revolve slowly about their centers, there will be no sudden change of position of any part of the dam to receive any shock. The flow will be stopped gradually and there will be a smoothness of operation. It is expected that the dam will not be raised until the upper part of the canal has been entirely clear of any floating craft that may have been moored in the canal.

(e) This structure can be built largely of standard structural sections. The dam proper is composed of plate girders. The anchors for holding the dam are of H-sections, channels, angles, and eyebars. The hinges are of steel plates, continuous on the downstream side the full width of the lock. These hinges are riveted to the flanges of the plate girders with the hinge pins in place; the pins may be made in sections of any length. The anchor hinges on the upstream side, holding the dam, are riveted to the webs of the plate girders. All the hinge connections have rivets in transverse or direct shear. It is not considered necessary that these hinges be bushed because movement of the parts is slow and infrequent. The upstream flanges of the plate girders consist of angles having their meeting legs machined, forming a close and watertight joint. The lower girder is provided with a box section, one side of which, when the dam is raised, abuts against a rubber sill, making a watertight joint.

(f) The flexible shutter dam is placed within the lock enclosure, between the upper guard gate and the upper operating gate at the elevation of the canal bottom of the upper approach. When the lock is unwatered it can be completely inspected, painted, and repaired, if required.

(g) As all parts will be placed out of view, the proposed dam has no objectionable features from an appearance point of view.

(h) Because the structure is submerged, it is not a likely subject for enemy attack. The power plant can also be placed out of view, if desired.

(i) As the flexible shutter dam is light in weight, compared with other types of movable dams, the cost will be considerably less. Cost of maintenance is small because of the smaller number of parts and the ease of access and inspection when the lock is unwatered. The weights (in pounds) of parts of the proposed dam (lock 80 ft wide and 30 ft deep) are estimated approximately as follows:

Dam proper, including girders, connections, and half of anchors	310,000
Other half of anchors (supported on lock floor).....	53,000
Wire rope and sockets.....	7,000
Embedded anchorages.....	40,000
Contingencies.....	15,000
Total.....	425,000

The weight of two hoisting engines has not been included in the foregoing estimate. Because of abnormal wartime conditions, no estimate of cost is presented. Under normal conditions the average cost of steel and iron in the structure should not exceed 15¢ per lb.

#### SUMMARY

Probably no engineering structure has been given more study in connection with the operation of locks than a movable dam. Should a lock gate be wrecked by collision or otherwise while restraining the waters in a canal, sudden release of the water would cause a wave that would probably result in carrying away all vessels moored at the canal walls, and such vessels would be subject to collision below the locks and be wrecked. The magnitude of the catastrophe would depend on the head depth and size of the lock and canal. Having a movable dam will not avert the wrecking of a lock gate, but such dam will serve to stop the flow of water, permitting repairs to be made and restoring the lock for service.

The importance and necessity of having a movable dam is evident. The task of stopping a flow of water in a ship canal without such a structure would be tremendous.

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# AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

## REPORTS

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### ORGANIZATION, FINANCING, AND ADMINISTRATION OF SANI- TARY DISTRICTS

#### PROGRESS REPORT OF THE COMMITTEE OF THE SANITARY ENGINEERING DIVISION

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##### INTRODUCTION

1. *Preamble.*—This progress report is intended to stimulate discussion. The objective of the committee is to formulate the fundamental concepts of the subject assigned to it so as to provide a basis for the preparation of state enabling acts or for the amendment of existing enabling acts regulating the organization, the financing, and the administration of sanitary districts. It is hoped by the committee that there will be an opportunity during 1945 to receive and consider discussions of their progress report by engineers, administrative officers, and lawyers, and to present a final report in January, 1946, that will be useful to the Local Sections of the Society.

2. *General Statement.*—A sanitary district is a local governmental unit created under a state enabling act to finance, construct, and operate works for the collection and treatment of sewage, including industrial wastes amenable to joint treatment, and to provide adequate final disposal of the effluent and of the by-products of the treatment processes. The principal parts of the works to be built and administered by a sanitary district include:

- (a) House connections, mainly from the street sewer to the curb;
- (b) Sewers (lateral or collecting, trunk, and intercepting);
- (c) Sewage pumping stations; and
- (d) Sewage treatment works.

The three parts of the title—organization, financing, and administration—are interrelated. All spring initially from state enabling acts which give wide or only limited powers to the administering bodies that they create. Sanitary districts differ in size and in their objectives. Some undertake only the construction of intercepting sewers, main pumping stations, and treatment works and may serve a considerable number of municipalities. Some are coextensive with an existing city. A few include lateral or collecting sewers

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NOTE.—Written comments are invited for immediate publication; to insure publication the last discussion should be submitted by July 1, 1945.

in their construction, whereas others take over existing collecting sewers for operation and extension, or leave all of this to existing municipalities within the boundaries of the sanitary district. Thus, local conditions and practices are important and will affect many procedures as to administration and finance. Such physical and social differences explain in part the different procedures that have been adopted in the United States.

In the past, local governmental units have been given the power to accomplish one or all of these objectives and these units have been created as taxing units with the power, in general, to levy ad valorem taxes and special assessments. This grouping would include sanitary districts with taxing power and, in some cases, with power to collect charges for services only, the taxing power being denied. In a few instances these powers have been combined. The scope of legislative action is limited or prescribed, in some instances, by constitutional limitations upon the power to issue bonds and upon provisions relating to the levy and collection of taxes and special assessments.

3. *Present Status.*—There is a considerable difference of thought and practice regarding these matters by courts, commissions, governmental authorities, bankers, lawyers, and practicing engineers. Some aspects of these topics are relatively new and sound precedents are yet to be established. In some cases, well-established practices cannot or should not be disturbed. The many different enabling acts and methods of financing are ample evidence of this situation. Where competent professional service has been used, the design, construction, and operation of sewage disposal projects in sanitary districts have been ably accomplished.

The fundamental considerations leading to the best methods of organizing, financing, and administering sanitary districts are the subject of this report. The present status of sanitary districts, as a matter of historical record, is found in several publications, and reference is therefore made to the Bibliography in Appendix I.

## PART I. ORGANIZATION

4. *Fundamental Considerations.*—Certain steps are fundamental, and are determining factors, as regards the organization of a sanitary district, as expressed in the following questions:

- (a) How shall the district be originated and created?
- (b) How shall the boundaries of the district be determined?
- (c) How shall the administrative board be formed?
- (d) How shall the district be financed?
- (e) What shall be the powers of the district?

The answers to these questions constitute the fundamental steps which more or less control the preparation of an enabling act. In general, the committee feels that the steps required to create, form, and organize a sanitary district should be taken within the locality that will comprise the sanitary district.

5. *Origination and Creation.*—Under existing enabling acts, a sanitary district is generally originated and created in one of the following ways:

- (a) By a petition from a certain number (often 100 to 500) qualified voters or persons in the locality;
- (b) By the action of one or more existing local governing bodies, such as city or village councils, county supervisors or township boards; or
- (c) By the action of the state through its legislature or some designated department of the state, such as the Department of Public Health.

The committee favors the origination and creation of sanitary districts by local petition followed by a general vote by qualified persons within the designated boundaries. The petition should be presented to a duly elected local official or governing body who shall consider it, hold public hearings, approve or disapprove the action, and conduct the vote. The emphasis in the first place should be on local action.

In cases where the need for a sanitary district is great and the locality does not make sufficient progress, the state should notify the locality of this fact and—after a reasonable period of time, perhaps one year—should create the district under specific authority granted by legislative action.

6. *Extent of Sanitary Districts.*—The boundary of a sanitary district is important. Districts having the following extents are found:

- (a) Part of a municipality;
- (b) An entire single municipality, generally a city;
- (c) A single municipality and some adjoining unincorporated area;
- (d) A group of municipalities, generally adjoining, with some unincorporated area, the whole constituting a metropolitan area;
- (e) A single county or an area within a county;
- (f) An area within an arbitrary line, comprising a region, and
- (g) A river valley or drainage area.

The committee favors the determination of the extent by a competent engineer or board of engineers after an adequate survey and investigation of the locality. The expenses of this survey and other preliminary costs preferably should be appropriated by the existing local governing bodies affected. If the general public health is sufficiently involved, funds should be appropriated for part or all of the costs by the state. These costs may later be repaid by the sanitary district if and when it is created, organized, and financed.

If it becomes necessary for the state to create the district, the legislature should appropriate moneys for the preliminary engineering and administrative costs.

7. *Administrative Boards.*—Enabling acts provide several methods of setting up the administrative board, among which are the following:

- (a) Election;
- (b) Appointment (by an existing local governing body or person such as a local judge, a mayor, or a municipal council; or by the governor of the state); and
- (c) By designation in the enabling act as, for instance, the appointment of certain existing local officers otherwise elected.

Some enabling acts provide for a combination of the foregoing methods. The committee favors, in general, the election locally of the members of the administrative board. Many appointed boards have been eminently successful, however, and the appointment of board members by some locally elected official may be an acceptable method. Where several municipalities are included, the chairman or president of the sanitary district might be appointed by the governor from candidates not residing in the district. Experience also indicates that, except in very large districts, the board members should receive only nominal compensation. In very large or special districts the chairman of the board should receive an adequate salary for full-time service and, in some cases, each member should be so compensated. A board of three to five members is considered to be workable.

8. *District Financing*.—When a district is created, it should automatically collect a charge on the area within its boundaries to cover the reimbursement of the preliminary expenses and the payment of the administrative, engineering, and other expenses of the district prior to the receipt of moneys through the method of financing finally adopted.

9. *Powers of a District*.—The powers given to a sanitary district by the enabling act are not unusual; they are details of secondary importance to the fundamental considerations. The district should have the power to:

- (a) Make rules and regulations for the use of its facilities;
- (b) Join with other districts and municipalities in the exercise of common powers;
- (c) Contract with others to provide or receive sewage disposal service;
- (d) Prescribe methods for the disposal of objectionable industrial wastes not amenable to treatment with domestic sewage, and regulate the discharge of other industrial wastes;
- (e) Sell or otherwise dispose of effluent, sludge, or other by-products;
- (f) Require the use of its facilities whenever they are accessible;
- (g) Make and apply rates of charge for sewage disposal service and enforce penalties for non-payment;
- (h) Make use of suitable patented processes necessary to the work of the district; and
- (i) Purchase material, supplies, and equipment without competitive or formal bids up to a stated total cost for each transaction, the amount to be larger for large districts than for small districts.

## PART II. FINANCE

10. *Fundamental Considerations*.—A sanitary district must be able to raise enough money to cover all of its proper costs, including construction, operation, maintenance, and administration. These can be considered as the total annual cost, which includes debt service and all the current expenses of operation. This part of the subject reaches into the fundamental equities of the allocation of the over-all costs of providing sewerage or sewage disposal facilities. No single method in the United States has received predominant support by courts and commissions. A variety of methods and combinations of methods can

be cited. This diversity is caused, in part, by the historical aspects of methods of financing sewage disposal projects as follows:

(a) One of the earliest methods was to raise all necessary funds by taxes on property. This was accomplished by the issuance of general obligation bonds supported by property taxes and by an annual levy to support operating and maintenance costs.

(b) A second method in point of time has been to raise construction costs by taxes on property and operating costs by charges for sewage disposal service. This method has been used to a considerable extent in Ohio and will be found to apply to many present sewage disposal projects.

(c) A more recent method has been to raise all necessary funds by charges to persons or property receiving service. The distinguishing characteristic of this method is that the entire cost of the sewage disposal service is allocated to persons or property (or both) on whose account it is provided. The effort is made to allocate the total annual cost on the basis of the cost of providing the service.

(d) For many years, the cost of constructing sewers has been met by a frontage assessment, such as a charge per front foot of benefited property.

Methods of financing to be adopted should be considered in the light of local conditions, such as established methods and habits, public relations, convenience of administration, enforceability of collections, and the security of the revenue from year to year. The extent of the work to be built and operated will also influence the method of financing. Thus, practice and experience are pertinent items.

The foregoing methods of financing have different results as regards the fairness of the charges. If it has been or is fair to raise all of the costs through a general tax on real property, it is not generally fair to raise all of these same funds by a charge against direct users; that is, those having actual connections with the sewage disposal facilities. In some districts, such as the Brockton (Mass.) and the Buffalo (N. Y.) sanitary districts, and the Washington Suburban Sanitary District in Maryland, the methods of financing seem to approach equity in that an attempt is made to allocate the costs to persons and property reasonably in accordance with the costs of providing the service.

As between different methods, the sums involved should be considered. A total average annual cost of \$5.00 per capita amounts to about 1.4¢ per capita per day. This is a small unit and in some cases a moderate departure from the fairest charge in favor of a well-established method convenient to administer seems to be tolerable in practice.

Methods of financing in present use might be tested to determine how closely they result in an equitable distribution of the charges necessary to defray the total annual costs. It seems unlikely that an altogether equitable method of financing can be set up that is not too complicated, and consideration must be given to what is convenient and expedient. It is believed, however, that substantial equity can be accomplished through methods of financing that are not too complex.

An excellent statement on these matters, applicable to a municipality and including collecting sewers as well as intercepting sewers and sewage treatment works, is taken from an unpublished report by Frank A. Barbour, M. Am. Soc. C. E., as follows:

"\* \* \* General taxes are based on the valuation of property or on the ability of the individual to pay, regardless of the particular benefit derived by him from the expenditure of the money so obtained. Thus the cost of education, of policing of the community, the support of the poor and other public necessities, the benefit of which to the individual cannot be definitely estimated, is reasonably provided for. An assessment, on the other hand, is levied in proportion to the particular benefit accruing to individual parcels of land by the construction of a certain public utility. Unlike a tax it does not take something without any assurance of compensation but rather is intended to express, with as much accuracy as possible, the value added to the property by the works to meet the cost for which it is levied. An assessment implies permanency in the improvement and in the value added to the property. Obviously it must be apportioned on some logical, uniform and unchangeable basis such as the physical dimensions of the property and the rate per unit once established cannot be altered from time to time. The value added to the property may be merely potential and does not imply actual use of the public utility. Such use will constitute a further benefit which in amount may be estimated by rental charges based on units involving the elements of quantity and time and which are not necessarily fixed in amount as in the case of assessments.

"Having thus called attention to the essential characteristics of general taxation, assessments and rentals, it remains to be said that various combinations of these methods of raising money for sewer purposes have been used in different municipalities and, as will be shown later, with statutory authority.

"The construction of a sewerage system results in a general hygienic betterment to the community from which must follow an indirect return to all citizens regardless of actual use or opportunity to use the system. Further, in all sewerage works a portion of the expense for outfalls and purification works is due to topographical conditions or sanitary obligations to neighboring communities, which constitute a disability to be borne in some considerable part by the Town as a whole. It therefore follows that a portion of the money necessary for the installation of a sewerage system is logically raised through the general tax rate and, as will be seen by reference to the statutes, the law requires that at least one-quarter ( $1/4$ ) and not more than two-thirds ( $2/3$ ) of the total cost of the system must be so obtained. This is equivalent to a statement that at least one-third ( $1/3$ ) must be collected on some other basis than through the general tax. The problem is to determine what proportion of the whole cost between one-third ( $1/3$ ) and three-quarters ( $3/4$ ) shall be thus raised and how can the amount to be so raised be most equitably apportioned. Obviously the answer will be found in that method which will most fairly express the relative benefits derived from the system by the individuals taxed.

"The construction of a sewer in a street constitutes a substantial increment to the value of the abutting property—irrespective of how this property may be at present developed and aside from any actual use which may be made of the sewer. This betterment pertains to the land and can be best expressed on the basis of physical units of frontage or area within a certain distance of the street or a combination of the two.

"The greatest benefit from the construction of sewers will be conferred on those who actually connect with and use the system to dispose of their

daily wastes. In amount this benefit is logically indicated by the quantity of sewage discharged from the individual premises and most equitably expressed by rental charges based on the meter records of the water department or, where private water supplies are in use, on estimates or gaugings of the amount of liquid contributed to the sewer. It therefore appears that an equitable apportionment must recognize benefits in three degrees:—first, the general betterment resulting to the community in improved sanitation; second, the increased value to abutting property following the construction of a sewer in any street and third, and greatest, the benefit derived from actual use. Accepting these conclusions it follows that the method of raising the required income must include a part through the general tax levy, a part by assessment on land and a part by rental. In what proportion the total shall be thus divided remains to be determined."

Two comments on the foregoing abstract are of interest:

(a) An assessment rate per unit, once established, cannot be altered. It is a directive to be applied to a specific project and not as a general permanent charge throughout a sanitary district.

(b) The relation between the benefit from the construction of sewers to those actually connected and to the municipality as a whole should not in all cases be subject to the interpretation of the term "greatest benefit" as including all or practically all of the benefit. In so far as the sewers contribute to general sanitation and public health, the public benefit is a factor.

11. *Kinds of Service Provided by a Sanitary District.*—The kinds of service provided by a district depend upon local conditions, such as whether separate or combined sewers are used or whether there is a large amount of drainage through infiltration. Differences in the extent of the service are also pertinent especially as to whether or not collecting sewers are included. As a basis for discussing the problem of a fair distribution of the costs of sewage disposal service or of providing for the use of the facilities, the following items seem to be pertinent:

(a) The cost of structures and equipment needed to provide capacity for: The initial population, future growth, storm water in combined sewer systems, illegal storm-water connections, ground-water infiltration, and industrial wastes;

(b) The cost of operation, maintenance, and administration needed to permit the use of the sewage disposal facilities; and

(c) The cost or value of the service given to protect the public health and for the general welfare of the locality.

The problem is to find a fair and reasonable relation between the costs on the one hand, and the service and use on the other. Some of the foregoing cost items provide service mainly to area and others mainly to users. Thus the costs of providing service for storm water, ground water, and future growth may, in some cases, apply to, and depend on, area more than on direct use. For a particular district, a careful computation of the costs of providing these several services and uses may permit the formulation of an equitable schedule of charges. Thus the total annual cost of facilities to serve only the present

population as compared to the total annual cost of the usual facilities which include capacity for future growth will indicate the cost of providing for future growth. This might properly be charged to area, or at least to presently unoccupied area. The cost of handling storm water and ground water might also be estimated and this cost regarded as a measure of service to area. The remaining operating costs would be a charge on users.

The service given to protect the public health and for the general welfare of the locality is broad and general, not subject to actual computation. This is a real service, however, much of it being to users but some of it to property. Perhaps a reasonable percentage of the total annual cost can be allocated to this item.

The application differs in accordance with local conditions. Thus, a sanitary district comprising a relatively new sparsely settled area will justify and permit a wider application to area than an older well-built-up area.

12. *Rate Schedules.*—There are many different kinds of rate schedules in use. Some 250 schedules are admirably summarized in the Third Progress Report of the Committee on Sewer Rental Laws and Procedure of the Sanitary Engineering Division (6c).<sup>1</sup>

Rate schedules are adopted to provide for the collection of the needed revenues in accordance with the general method of financing so as to avoid unreasonable discrimination. This is a difficult task, not readily accomplished. Even in water rate schedules, which have a much broader basis of experience, there is a marked difference as, for instance, in the number of brackets, the slide or spread between small and large users, and the amount allocated to the public through fire protection service.

13. *Illustrative Schedule of Charges.*—For illustrative purposes only, a computation of a schedule of charges, rates or rentals, is presented, as follows:

Distribution of "Area Units."—

Item	Description	Area units
1	Residential (4,800 acres).....	2,090,000
	Fronting on a sewer line (85% of Item 1):	
2	Parcels smaller than 7,500 sq ft (95%).....	1,690,000
3	Parcels larger than 7,500 sq ft (5%).....	90,000
4	Total, Items 2 and 3.....	1,780,000
5	Not fronting on a sewer line (15% of Item 1)...	310,000
6	Total, Items 4 and 5.....	2,090,000
7	Business and industrial (720 acres).....	314,000
8	Fronting on a sewer line (70% of Item 7).....	220,000
9	Not fronting on a sewer line (30% of Item 7)...	94,000
10	Total, Items 8 and 9.....	314,000
11	Total area units (Items 1 and 7).....	2,404,000

<sup>1</sup> Numerals in parentheses, thus: (6c), refer to corresponding items in the Bibliography (see Appendix I).

## Number of Water Connections or Consumers.—

Item	Description	Number
12	Minimum number of consumers.....	12,500
	Minimum Number of Intermediate Users:	
13	Mostly residential.....	9,700
14	Large users.....	2,800
15	Total, Items 12, 13, and 14.....	<u>25,000</u>

## Distribution of Revenue and Cost.—

Item	Description	Dollars
	Area Distribution of Revenue:	
	By "Area Units" Fronting on a Sewer Line—	
16	Small parcels in residential zones (1,690,000 units at \$0.10).....	169,000
17	Large parcels in residential zones (90,000 units at \$0.05).....	4,500
18	Business and industrial zones (220,000 units at \$0.15).....	33,000
19	Total, Items 16, 17, and 18.....	<u>206,500</u>
	By "Area Units" Not Fronting on a Sewer Line—	
20	Residential zones (310,000 units at \$0.03)....	9,300
21	Business and industrial zones (94,000 units at \$0.06).....	5,600
22	Total, Items 20 and 21.....	<u>14,900</u>
23	Total revenue by area distribution (Items 19 and 22).....	221,400
	Use Distribution of Revenue:	
24	Minimum users (Item 12), 12,500 at \$6.00.....	75,000
25	Intermediate users (Item 13), 9,700 at \$12.00...	116,400
26	Large users (Item 14), 2,800 at \$42.00.....	117,600
27	Special industrial wastes.....	15,600
28	Contracts with adjacent areas.....	12,000
29	Total revenue by use distribution (Items 24 to 28)	<u>336,600</u>
30	Total revenue (Items 23 and 29).....	558,000
31	Less 10.4% margin for delinquent accounts.....	58,000
32	Total estimated net revenue.....	<u>500,000</u>
	Annual Cost:	
33	40% in area distribution.....	200,000
34	60% in use distribution.....	300,000
35	Total estimated annual cost (see Item 31, Appendix II).....	<u>500,000</u>

This computation is related to several units of an hypothetical district given in Appendix II. The illustration applies more particularly to a relatively new sparsely settled area for which the sanitary district is to provide and administer the service of collecting sewage as well as treatment. The total annual cost of \$500,000 is assumed to be allocated to "area" and to "use" under the assumption that 40% is charged to area (Item 33) and 60% to use (Item 34). An "area unit" used in computing the charge to area is 100 sq ft. The charge to "use" is based on the water consumption assuming a completely metered water system. Some general similarity to methods of financing in use since about 1925 in the Washington Suburban Sanitary District will be noted.

The foregoing illustration is not fully related to established and sometimes convenient practice. Local conditions should be considered within reasonable limits of tolerance.

*14. Financing Rights under Enabling Acts.*—The enabling act, subject to the provisions of the state constitution and related court decisions, establishes the rights of a sanitary district as regards methods of financing. It either prescribes a method of financing or instructs the board to use an equitable method. The committee favors a broad general grant of power rather than a specific direction. Thus, the financing of a district might be by moneys collected from the occupants or owners (or both) of all properties served by the sewage disposal facilities in the district, allocated by the governing body on a fair basis and by a fair method.

*15. Governmental and Proprietary Aspects.*—The question has been raised as to whether sewage disposal and, in general, the operations of a sanitary district are governmental or proprietary functions. Throughout the forty-eight states, court decisions and legal opinions can be cited on both sides and in some cases the opinions seem to be conflicting. The committee is not able to state an opinion on this important aspect. It feels, however, that the administration of a sanitary district (because an important objective is to protect the public health) is more a governmental than a proprietary function.

Some cases seem to indicate that the method of financing depends on whether the governmental or the proprietary aspect applies. This may be illustrated as follows:

(a) Under the proprietary powers, the district would be financed by users and not by property. Under this assumption, it is held that sewage disposal is a business comprising the furnishing of a service (or possibly a commodity in reverse) for which users only should pay. Under this application, sewage disposal would be regarded as a service similar in nature to garbage removal and disposal.

(b) Under the governmental powers, the district may be financed by any equitable allocation of the costs of providing sewerage or sewage disposal service, or both.

Further guidance is desired on this aspect of the subject assigned to the committee. Reference to a large number of court decisions and legal opinions is given in Appendix III.

## PART III. ADMINISTRATION

16. *General Statement.*—The administration of sanitary districts is largely a matter of detail and does not involve such fundamental considerations as enter into a discussion of organization and financing. Experience has amply demonstrated that the ability, character, and intelligence of the members of the board are highly important. Enabling acts in each state should make it likely that such local citizens will be willing to serve.

In general, the administrative board should not only perform its current tasks ably and with a high quality of professional work, but they should also look forward to increasing standards of sanitation, to improved methods of sewage treatment, and to the future needs of the district.

## SUMMARY

The request for wide discussion during 1945 denotes that many of the foregoing paragraphs should be considered as tentative. The committee hopes that, by the expression of the opinions and comments of others, they will be able later to present a useful final report.

Respectfully submitted,

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*Committee on Organization, Financing, and  
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January 18, 1945

## APPENDIX I

BIBLIOGRAPHY ON THE ORGANIZATION, FINANCE, AND ADMINISTRATION  
OF SANITARY DISTRICTS

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- (6) **Sewer Rental Laws and Procedure**, Committee of the Sanitary Engineering Division, Am. Soc. C. E.
- (a) First Progress Report, *Civil Engineering*, March, 1937, p. 220.
  - (b) Second Progress Report (unpublished), presented at January, 1941, Meeting.
  - (c) Third Progress Report, *Proceedings*, Am. Soc. C. E., November, 1942, p. 1585.
- (7) **"Governmental Liability in Torts,"** by E. M. Borchard, *Yale Law Journal*, Vol. 34, January, 1925, p. 235. (a) p. 324.
- (8) **"The Law of Municipal Corporations,"** by E. McQuillin, 2d Ed., 1937, Vol. 6, p. 1060. (a) p. 1058. (b) p. 1038.
- (9) **"Public Health Law,"** by J. A. Tobey, 2d Ed., 1939, pp. 275-276. (a) p. 277.

## APPENDIX II

### UNITS FOR A TYPICAL MUNICIPALITY WITH SEPARATE SEWERS

#### Statistical Data.—

Item	Description and unit	Quantity
Area, Distribution in Acres—		
1	Streets, 25%.....	2,000
2	Parks, 6%.....	480
3	Residential, 60%.....	4,800
4	Business and industrial, 9%.....	720
5	Total (12.5 sq miles).....	8,000
6	Population (12.5 per acre).....	100,000
7	Average annual sewage flow (million gallons daily) ..	10.0
8	Collecting sewers (million feet).....	1.2
Property Frontage (Million Linear Feet)—		
9	On sewer lines.....	1.8
10	Without sewer lines.....	0.2
11	Total frontage.....	2.0
Number of Properties—		
12	Residences (100,000 at 4.5 persons per residence) ..	22,200
13	Business and commercial properties.....	2,800
14	Total occupied properties.....	25,000
15	Land parcels of less than 7,500 sq ft.....	31,500
16	Land parcels larger than 7,500 sq ft.....	3,500
17	Total properties, including Item 14.....	35,000
18	Vacant parcels of land (Item 17 minus Item 14) ..	10,000

*Total Cost Data.—*

Item	Description	Thousand dollars
<b>Construction—</b>		
19	Collecting sewers.....	3,200
20	House connections to the street line.....	800
21	Intercepting sewers.....	1,000
22	Sewage treatment plant.....	1,000
23	Total cost of construction.....	<u>6,000</u>
<b>Annual Cost of Operation—</b>		
24	Administration.....	25
25	Collecting sewers.....	60
26	Intercepting sewers, pumping stations, and sewage treatment plant.....	75
27	Total operating cost (Items 24, 25, and 26) ..	160
<b>Annual Average Debt Service—</b>		
28	Collecting sewers and house connections.....	240
29	Intercepting sewers, pumping stations, and sewage treatment plants.....	100
30	Total debt service.....	340
31	Total annual operating cost (Item 27 plus Item 30)	<u>500</u>

*Unit Cost Data, Operation, and Debt Service.—*

Item	Description	Dollars per year
32	Per capita cost of operation and debt service (Item 31/Item 6).....	5.00
33	Per acre other than streets and parks (Item 31/Items 3 and 4).....	90.75
<b>Per Foot of Frontage—</b>		
34	On sewer lines (Item 31/Item 9).....	0.28
35	Total (Item 31/Item 11).....	0.25

## APPENDIX III

## GOVERNMENTAL AND PROPRIETARY FUNCTIONS OF MUNICIPALITIES

Sanitary districts in general (sometimes termed sewer districts, sanitation districts, etc.) are governmental agencies created usually under the authority of a state law for the purpose of constructing and maintaining sewer systems, or sewage treatment or disposal works, for the benefit of a particular limited area. In many cases they are organized to furnish sewerage in areas which are not included within an incorporated municipality. In some cases such districts may be organized as a means of joint sewage disposal by several municipalities with or without adjacent unincorporated territory. In other cases sewerage districts (usually of the special assessment type) may be organized by a municipi-

pality to furnish sewers to an unsewered area within its boundaries. There are many variations in type and purpose.

In general, such districts have many of the characteristics of municipal corporations, although their activities are circumscribed within a special field of service. In some jurisdictions they are designated to be municipal corporations, either by statute or by judicial decision. Usually, however, they are termed "quasi-municipal" corporations. Some of their functions appear to be rather definitely governmental in character, in a manner similar to that of municipalities.

There appears to be a wide difference in judicial decisions in the several states as to whether the functions of municipalities (and therefore of sanitary districts as quasi-municipal corporations) in relation to sewerage or sewage disposal are "governmental" or "proprietary" (also called "ministerial," or "corporate," or "municipal") in character. In general, governmental functions are those which are performed by the municipal or quasi-municipal corporation for the welfare of the state, and include such activities as police and fire protection, public health, and public school education. Proprietary functions, by contrast, are those which are performed primarily for the benefit of local inhabitants, and as a rule include such public works (frequently revenue producing) as water works, gas and electric plants, public markets, tourist camps, wharves, public transportation, and many others. Historically, most of such functions were originally performed by private individuals or corporations in the anticipation of profit. Whether the municipality or district derives a profit from such activities appears in some jurisdictions to be a factor in classifying it as proprietary; in other jurisdictions it does not appear to have any weight.

Not only are the court decisions at variance between states, but in some decisions there appears to be confusion of thought and logic within the decisions themselves. The reason for this variance is apparently due to the fact that many of the decisions result from suits to recover damages from municipalities. It appears to be a rule of law, based on ancient precedent extending back to the time when "the king can do no wrong," that the sovereign government cannot be sued by a private citizen, provided the function or act was "governmental" in character. As long as governments were relatively simple in their operations, no great confusion in thought occurred; but under today's conditions, with local, state, and national governments branching out into many fields of activity far outside of the early concepts of governmental functions, and furnishing services and materials formerly purveyed by individuals and corporations as a business, it has become less possible to discern clearly the distinctions between "governmental" and "proprietary" functions.

Some of the confusion must arise from a too narrowly circumscribed view of the purpose and effect of sewerage and sewage treatment, and from a lack of knowledge of the public health aspects of these matters. Courts have tended to look upon sewerage, in particular, as a matter undertaken and performed for purely local benefit only, and of little or no state-wide interest.

Nothing could be further from the facts under modern conditions of rapid transportation by train, automobile, and airplane, in a nation whose people in

normal times travel freely over considerable distances. Sanitary defects in even comparatively small communities thus present opportunities for the infection with communicable disease of a relatively large number of persons, who may in turn transmit such diseases by secondary infections over wide areas. It is only necessary to call attention to the epidemic of amoebic dysentery which was spread to nearly all states in the nation in 1933, as the result of comparatively minor defects in sanitation in two hotels in Chicago, Ill. This was merely a spectacular illustration on a large scale that defects in sanitation in any community are at least of state-wide, and at times of national, concern. The epidemiological records of any state will furnish many illustrations on smaller scales that communicable diseases resulting from unsanitary conditions in one local community are carried to other communities.

For this and other reasons, the sewerage of communities is a matter of public health concerning the state as a whole. It is not merely a local benefit to a particular area. It is directly and indirectly a matter of public health significance. In fact, that phase of sanitation which provides for the safe disposal of human excreta is a basic factor in public health. It is generally recognized by the courts that those activities of government which deal with public health are "governmental" in character, as contrasted with "proprietary" functions undertaken primarily for local convenience and benefit.

Furthermore, some of the confusion on this matter apparently arises from the fact that most of the decisions result from litigation involving suits for damage of some type. In many such cases the courts seem to have endeavored to escape from the concept of the governmental nature of sewerage and sewage disposal, as a phase of public health practice, by declaring sewerage and sewage disposal to be proprietary and local in character, in order that the injured party might recover damages from the governmental agency sued.

As a commentary on this situation, E. M. Borchard states (7):

"In the construction and maintenance of public works or improvements, such as sewers, drains, etc., we find a greater disposition to hold the city liable for negligence. The explanation of this disposition can hardly be found in the usually ascribed reason that such undertakings are not governmental, but ministerial in character; it is found rather in the fact that when a public enterprise creates a direct nuisance to private property, the governmental nature of the undertaking seems, in the minds of the courts, to become subordinate or immaterial. Certainly there is nothing less governmental in providing a city with sewers than in providing it with schools and fire protection, and the courts often find themselves involved in the problem of determining whether sewers and drains do not partake of means to safeguard the public health, usually regarded as a governmental function."

Furthermore, according to Mr. Borchard (7a):

"Immunity or liability in these matters depends upon the view of the particular courts as to whether the duty is governmental or corporate, and in this respect there is the usual wide divergence, depending often upon the particular line on which the courts in that jurisdiction got started. There is no discoverable operative principle."

The confusion of thought has been well illustrated by E. McQuillin (8):

"On the other hand, certain duties and functions are well settled as being corporate and not governmental, including the construction and maintenance of municipal water and light plants, \* \* \* the construction and repair of sewers, \* \* \* and generally the management of property owned by the municipality \* \* \*. Furthermore, any business conducted by the municipality for profit involves the exercise of corporate rather than governmental functions. \* \* \* And even if a duty is primarily a governmental one, it is converted into a corporate one when it is being conducted for profit \* \* \*."

Again, quoting Mr. McQuillin (8a):

"It is well settled that a municipal corporation is not responsible for the negligent acts of its employees who are endeavoring to carry out the regulations of the city to promote the public health and to care for the sick and destitute. In the collection and disposition of garbage, undoubtedly the city acts for the public health and discharges a governmental function."

Rather obviously, if Mr. McQuillin admits that in the collection and disposal of garbage the city "undoubtedly discharges a governmental function," his argument for the proprietary nature of sewerage fails, for the reason that sewerage and sewage disposal are much more closely related to the public health than is garbage collection and disposal.

Further illustration may be had by reference to the views of J. A. Tobey (9):

"Although the proper and safe disposal of sewage is recognized as an important health measure, it is a well-established rule of law in this country that a municipal corporation is liable to individuals for nuisances caused by the disposal of its sewage, since this is a corporate and not a government duty \* \* \*."

"An exception to this rule is in the case of the discharge of sewage into tidal waters."

Furthermore, quoting Mr. Tobey (9a):

"By the weight of legal authority, nevertheless, the construction and institution of a municipal sewer system is a governmental function, but its operation and upkeep is a proprietary function."

It may well be that a city is liable for nuisances created by sewage disposal, although not necessarily on the grounds stated by Mr. Tobey, but to state that the construction of sewers is governmental in character, whereas the operation of a sewerage system is proprietary in character, appears illogical. The mere existence of a sewer system in the streets would be of no value whatever to the public health; only when the system is connected to premises, and in actual operation, do actual benefits to the public health occur.

Another fallacy is implied in Mr. Tobey's statement—that there is no liability for nuisance when sewage is discharged into tidal waters. The extensive fouling of the beaches along Santa Monica Bay by the inadequately treated sewage of Los Angeles, Calif.; the strong smells along the easterly and southerly shores of San Francisco Bay, in California, resulting from the discharge of raw

sewage at the shore line; and some of the conditions in other harbors, need only to be mentioned as examples to dispose of this fallacy effectively.

The committee has not attempted to make an exhaustive search for court decisions bearing upon either the matter of "governmental" as contrasted with "proprietary" aspects of sewerage and sewage disposal, or the status of sanitary districts as "quasi-municipal" or "municipal" corporations. It is hoped that engineers from the several states will furnish data as to the leading decisions in their states, with statements as to the apparent trend of judicial opinion. Suffice it to state that the decisions vary widely, and only a few will be quoted or cited, mainly for the purpose of illustrating contrasts.

Perhaps two quite similar cases, in which diametrically opposed decisions appear to have been rendered, will serve as the first illustration. In both cases, a child had drowned by falling into an open tank at a sewage plant, which was not protected by fencing, and the parent sued the city for damages. In the case of *Barker vs. City of Santa Fe* (136 Pacific 2d 480), April 14, 1943, the New Mexico court ruled that the construction and repair of sewerage facilities was not a governmental function, and that damages could be recovered; but in *Gatcher vs. City of Farmersville* (151 S.W. 2d 565), May 7, 1941, the Texas court ruled that sewage disposal was a governmental function precluding the recovery of damages.

In the New York case of *Hughes vs. Auburn* (55 N.E. 389) in 1899, it was ruled that in the construction and maintenance of a sewer or drainage system a municipal corporation exercises part of the governmental powers of the state.

In the South Carolina case of *Dillon Catfish Drainage District vs. Bank of Dillon* (141 S.E. 274), January 9, 1928, it was held that the civil code provided for the creation of drainage districts to promote agriculture, public health, convenience, etc., which is a "well-recognized governmental function."

In the Georgia case of *City Council of Augusta vs. Cleveland* (98 S.E. 345), February 13, 1919, the court said (page 346):

"We are of the opinion that the duty of a city to maintain its sewerage drainage system in a good working order and sanitary condition is a governmental function. That such maintenance is connected with, and has reference to, the preservation of the public health is so well known and so generally recognized that courts will take judicial cognizance thereof."

In the Kansas case of *Foster vs. Capital City Gas and Electric Company* (265 Pacific 81), March 10, 1928, it was ruled that the construction and maintenance of a sewer system is ordinarily a governmental function.

In a Texas case, *City of Gladeswater vs. Evans* (116 S.W. 2d, 486), March 10, 1938, it was stated that a municipality, in building a storm sewer, was engaged in a governmental function. In the North Dakota case of *Hamilton vs. City of Bismarck* (300 N.W. 631), November 5, 1941, it was also held that, in constructing a sewer system, the city was acting in its governmental capacity.

However, in a Pennsylvania case involving sewer rentals (*Petition of City of Philadelphia*, 16 Atlantic 2d 32, October 28, 1940), it was ruled that sewers, like water systems, are owned and operated by municipalities in their proprietary capacity and not in their governmental capacity.

In Ohio, in *Union Properties vs. City of Cleveland* (49 N.E. 2d 571), April 12, 1943, it was decided that the providing of sewers and sewage facilities is in the nature of a governmental function.

In Wisconsin, in the case of *Hasslinger vs. Village of Hartland* (290 N.W. 647), March 12, 1940, it was ruled that the operation of a sewage disposal plant is a governmental function.

Courts have used the matter of revenue as a criterion for the determination of whether a municipal function was or was not proprietary. Mr. McQuillin (8b) states:

"A municipal government has a double function, first, the private, proprietary function, and second, the legal duties, from which the municipality obtains no revenue or other special benefit in its corporate capacity."

In the Louisiana case of *Floyes, for Use and Benefit of Floyes, vs. City of Monroe* (194 Southern 102), March 4, 1940, it was stated that the pivotal fact in determining the exercise of a governmental function was whether it was done purely as a public duty, or for pecuniary profit. Directly contrary to this was the Georgia decision in *Watkins vs. City of Toccoa* (189 S.E. 270), September 18, 1936, where it was held that the manufacture and sale of sanitary toilets by the city was a governmental function, and the fact that the city derived a profit thereon did not make it liable for negligence.

The manner in which revenue for the maintenance and operation of sewerage facilities is obtained, perhaps, has no real bearing upon the determination of whether sewerage is a governmental or a proprietary function of a city or a district. The cost of sewerage may be defrayed either wholly or in part from taxation or from usage charges, without affecting the real nature of the function as a phase of public health protection of both the state and the local community. Whether or not a profit may be realized from such taxes or charges appears somewhat to be academic, for the following reasons: (a) From the nature of the service, a true profit cannot be obtained in actuality; and (b) in practice, it is extremely doubtful if electors would permit the imposition of excessive charges or taxes which would result in an apparent profit.

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# AMERICAN SOCIETY OF CIVIL ENGINEERS

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## DISCUSSIONS

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### STORAGE AND THE UNIT HYDROGRAPH

#### Discussion

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BY C. O. CLARK

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C. O. CLARK,<sup>39</sup> JUN. AM. SOC. C. E.<sup>39a</sup>—The discussions represent much thought and labor at a time when most of the writers are actively engaged in entirely different matters. The expressions which corroborate the paper are gratifying and greatly appreciated. The divergent views outline some fertile fields for further investigation of problems which are still largely dependent on judgment and should stir stimulating and constructive thinking. Some of the queries and counterproposals presented lie beyond the ability of the writer to answer; some have been answered by other discussers; and a few are reviewed in the closing discussion. All are sincerely appreciated.

Mr. Sweet pertinently emphasizes that the labor involved in unit hydrograph and flood-routing techniques is excessive for routine river forecasting. Reduction of this labor by the development of simplified empirical coefficients is necessary and highly desirable. However, the simplification might be based upon a large number of hypothetical floods computed by appropriate methods as well as upon records of past floods. As Mr. Sweet states regarding the James River, nonuniformity of storm distribution may produce variable stage relationships between the gages along the river. If records of only a few large floods are available, it is easy to assume that relationships are fixed, only to be embarrassed when nature proves otherwise. However, unit hydrograph study of hypothetical adverse distributions of storm rainfall will reveal many of the variable possibilities which may serve as guides for good forecasts rather than as alibis for poor ones. The James River, with its two halves on either side of the Blue Ridge Mountains in Virginia, offers opportunities for illuminating study to any one who believes that nonuniformity of runoff distribution on an area is not significant, or that gage-to-gage stage relationships are simple or constant.

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NOTE.—This paper by C. O. Clark was published in November, 1943, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: February, 1944, by James S. Sweet, and Otto H. Meyer; March, 1944, by L. K. Sherman, Gordon R. Williams, and Ray K. Linsley, Jr.; May, 1944, by E. F. Brater, L. C. Crawford, Robert E. Kennedy, and Victor H. Cochrane; June, 1944, by Franklin F. Snyder; September, 1944, by Howard M. Turner; and December, 1944, by Don Johnstone.

<sup>39</sup> Hydr. Engr., U. S. Engr. Office, Winchester, Va.

<sup>39a</sup> Received by the Secretary January 12, 1945.

Major Meyer clarifies several points in the paper, and appropriately warns of situations in which  $K$  is not a constant. Unit hydrograph theories and constant time elements of flow are convenient oversimplifications of a most complex form of open channel flow. They are by no means universally applicable. The paper treats only the elements common to unit hydrograph theory and to the theories of flood routing based upon constant values of  $K$ . "Lag routing," to which both Major Meyer and Mr. Snyder refer, is a very practical and expeditious method of flood routing. Fig. 3 and its supporting discussions demonstrate the essential agreement between this method and the much more laborious application of the Muskingum technique. The construction of the time-area concentration curve of Fig. 8 and its routing modification conform essentially to "lag routing" procedure.

Lag routing is an empirical approximation of some solutions of the storage equation; it is not itself a solution of the storage equation for any fixed relationship between storage capacity and discharge. It is an expedient method for use in a natural state of river development, as in flood forecasting or in computation of past floods. However, the writer would be unable to determine, with it, the effect of improvements which change the storage capacity along a river, such as extensive levee systems—or particularly those improvements which change the relative influence of inflow upon storage, such as a series of dams.

Thus, for solving routing problems which involve a change in storage-discharge relationships, the writer believes that the storage equation will be more reliable than empirical "lag routing." As Major Meyer writes, there are several methods other than the Muskingum method for the solution of Eqs. 2, 4, and 5.

Mr. Sherman—the creator of the unit hydrograph—reviews the underlying theory of this excellent hydrologic tool with masterliness. Possibly through misinterpretation of the bar pattern of Fig. 8 and through the assumption that curve  $a$  is a net rainfall pattern of twelve unit periods rather than a representation of shape of drainage area, Mr. Sherman concludes that the resulting derived hydrograph was not a unit hydrograph. In Fig. 8, a bar pattern representing net rainfall would be of infinitesimal duration at zero time. In Fig. 9, rainfall was principally confined to that on April 25 and that early on April 26.

An apology is made to Mr. Sherman and to others who found difficulty in checking the computations in Table 1, particularly the conversion of Col. 8 to Col. 9. The conversion at a rate of 69,000 (cu ft per sec)-half-days per in. is a rounded value appropriate to a drainage area of 1,300 sq miles, rather than to one of 1,335 sq miles. The former area is that for which the first derivation on this stream, determined for a dam site above the gage, was prepared. Intent upon presenting an unretouched, original, routine solution, the writer failed to notice and properly explain the small difference in drainage area. The routing calculation illustrated is for 15 hours, as stated in Table 1, rather than for 9 hours, as stated in the text. Subsequent investigations indicated the smaller value to be more suitable. There are also some slight errors in Table 1, all well within the range of acceptable routine office practise, and

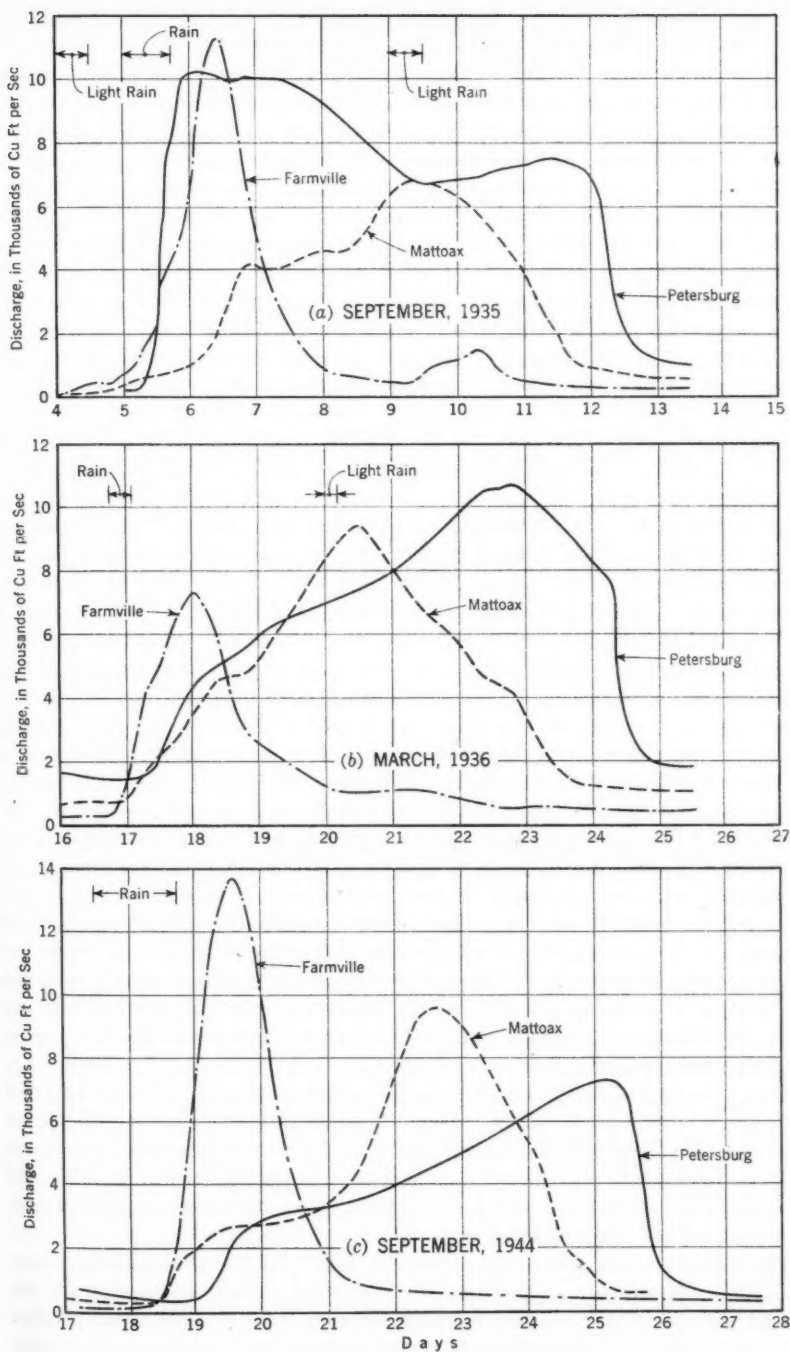


FIG. 18.—HYDROGRAPHS OF THE APPOMATTOX RIVER

there is much room for improvement in the analysis. The entire original analysis was prepared in 4 hours and was not then intended as a part of a published paper. The presentation of these data was intentional. The real value of a hydrologic tool is much better judged by the rough-hewn original product with its faults, than by the prettier refinements which most investigators possess and ultimately adopt for use.

Mr. Sherman comments on the size of watersheds to which unitgraphs may be applied. He correctly states that size is not a barrier to the adequate treatment of nonuniformity of rainfall with respect to time. On the other hand, nonuniformity with respect to areal distribution of the rainfall creates variations such as those shown in Fig. 18. Each of these hydrographs resulted from a rainfall brief enough to be considered a unit rain. The relative size of hydrographs at the three stations (gaging, respectively, 23%, 55%, and 100% of the Appomattox River above Petersburg) and rainfall records indicate heavier rainfall on one end or the other in each case. Of the hydrographs at Petersburg, none would be suitable for derivation of a unitgraph by any method which assumed that the hydrograph resulted from a uniform distribution of runoff. In the proposed method, which makes no such assumption, an equally suitable unitgraph could be derived from each of the records. However, neither of the hydrographs at Petersburg could be reproduced well by use of a single unit hydrograph, because, although a reproduction would represent flow from uniformly distributed runoff, the record does not. Subdivision of the area and the use of three component unitgraphs like those in Table 2 produce better results. The methodology as extended by Mr. Turner, incorporating both area and runoff depth in the time-area curve, gives much better results.

The possibilities of storage acting to increase outflow drew varied comment from Messrs. Meyer, Williams, Linsley, Brater, Snyder, Turner, and Johnstone, ranging from "controvert[ing] accepted concepts and experience" and "erroneous" to "familiar to all persons who have worked with reservoirs storing appreciable volumes of water under the backwater profile." The writer agrees heartily with those who suggest "These conclusions should be examined critically." The possibility should not be passed over lightly, however, as failure to recognize the possibility leads directly to overoptimism about the effectiveness of certain types of flood control dams, and to underestimation of the importance of avoiding fixed-pool operations at power and navigation dams and unnecessarily rapid gate closures.

The writer's comments about the inflow-controlled storage were intended to be qualitatively explanatory (rather than quantitatively assertive) of the statement (see heading, "Valley Storage"); "Valley storage does not always decrease flood peaks \* \* \*." The excellent discussions reviewing the case against such a viewpoint appear to require some presentation of the data supporting an affirmative case.

The writer's use of a uniform channel in Fig. 1 for the qualitative explanation was ill chosen from a quantitative viewpoint. The uniform channel was selected because of the simple, familiar form of the two backwater profiles. Obviously, however, a uniform rectangular channel would not have an  $x$ -value in excess of 0.5. Therefore, Professor Brater's quantitative criticism is well

founded. A quantitative demonstration might consider that the rectangular channel possesses a non-water-carrying flood plain several times as wide as the channel at the upper end, converging to the channel limits at the lower end. By appropriate selection of the flood plain width at the upper end of such a channel (which would be quite similar to the channels of many natural streams carrying relatively small flows in their extensive overbank areas), the volume of stored water under the backwater curve could easily be made to exceed 720 acre-ft, which is the volume Professor Brater's analysis indicates as the limiting volume that could be passed without surge.

Channels—the variable cross sections of which effectively converge in the direction of flow—might be expected to possess storage capacities influenced more by inflow than by outflow. Such channels might include:

- (a) Channels converging in variable depth (such as low-slope channels approaching a steeper channel and thus flowing from a zone of great range in stage to one of small range in stage and at the same time possessing flood plains of about equal width, or such as reservoirs maintaining fixed levels at the dam with pronounced fluctuation at their upper ends, or such as reaches ending in tidal estuaries); and
- (b) Channels converging in width but of relatively constant range in flood rise.

In the first class may be found certain South Atlantic streams which cross the Piedmont Plateau flowing toward the Fall Line. The Fall Line is a cascade zone which lies between the Piedmont Plateau and the Atlantic Coastal Plain. For example, the Appomattox River, whose drainage area above Petersburg was shown in Fig. 6, lies on the Piedmont Plateau. Two gaging stations are located above the Petersburg gage: Mattoax, Va., with a drainage area of 729 sq miles, is on the 2.0-day isochrone of Fig. 6, and Farmville, Va., with a drainage area of 306 sq miles, is on the 4.3-day isochrone of Fig. 6.

The typical flood rise at Mattoax in feet is much larger than that at either Farmville or Petersburg although flood plain widths are comparable for the three stations. The variable cross section of flood flow can be considered diverging in the direction of flow from Farmville to Mattoax and converging from Mattoax to Petersburg. Hydrographs of flow at these three stations resulting from rainfall of short duration are shown in Fig. 18. (Note that the later portions of the hydrographs at Mattoax and Petersburg must be essentially the flow component from above Farmville.) To occupy such a position in the hydrographs shown, it appears that the flood wave might have undergone flattening from Farmville to Mattoax and then have been built up from Mattoax to Petersburg. A more provable indication of the type of storage influence represented by  $x$ -values greater than 0.5 is the increase in recessional slope of the Petersburg hydrograph as compared with the Mattoax hydrograph.

In the second class might be short reaches of many rivers, as most flood plains alternately widen and narrow. Many rivers have long reaches through which flood waves flow without appreciable reduction.<sup>40</sup> It seems a little more

<sup>40</sup> "The Floods of March 1936," *Water-Supply Paper No. 800*, U. S. Geological Survey, pp. 89 and 90 (last two hydrographs on each page).

rational to interpret such unreduced passage as the result of alternate slight increases and decreases in peak rather than as a passage of an unvaried peak. A record of a wave passage having slightly more increase than decrease was shown in Fig. 12, in which the hydrograph of flow at Danville appears to have built up slightly and shortened to become the later portion of the hydrograph at South Boston. The discharge from the intervening drainage area would have been insignificant on the twenty-first and twenty-second days.

Most rationalizations about the possibility of storage operating to build up peak flow turn upon the question: "Granting that a volumetric differential exists between the equilibrium profiles of flow applicable to flow distributions before and after a change in inflow, can the change from one profile to another take place rapidly enough to cause a surge at the downstream end?" Fig. 19

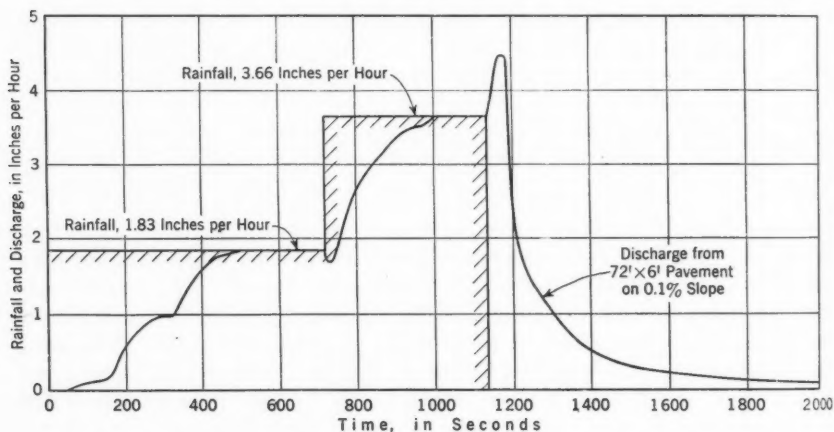


FIG. 19.—HYDROGRAPHS FROM EXPERIMENTS BY C. F. IZZARD AND M. T. AUGUSTINE

shows the experimental hydrographs which C. F. Izzard, Assoc. M. Am. Soc. C. E., and M. T. Augustine obtained by sprinkling a pavement.<sup>41</sup> In this case, the change from one flow distribution profile to another did take place rapidly enough to cause a surge. How fast does the change take place in rivers? How fast does storage volume react to a change in flow?

Several discussers mention the assumed instantaneous reaction of storage to a change in flow (which is characteristic of most flood-routing techniques and favorable to reduction of peak flow in the method discussed when  $x$  is less than 0.5) as the error which leads to the author's conclusion. The formula for reaction time presented by Messrs. Williams and Brater is quite favorably regarded by many, whereas Mr. Snyder appears to regard any value less than  $K$  as inadequate. Several intermediate viewpoints have some adherents. The time elements indicated by Eq. 17, first discussed by Mr. Williams, vary from very long values for a shallow depth at the beginning of a flood to crest-depth values which are very short in relation to the time elements of mean

<sup>41</sup> "Preliminary Report on Analysis of Runoff Resulting from Simulated Rainfall on a Paved Plot," by C. F. Izzard and M. T. Augustine, *Transactions, Am. Geophysical Union*, Vol. 24, 1943, Pt. I, p. 500.

flood-wave travel. Thus, in the reach of river to which Mr. Williams applied the formula, the channel depth was more than 50 ft at the time of crevassing. For this depth, the formula discussed would indicate less than 40 min. (Neither the methods of observing stage, nor those of defining the time of crevassing of a levee are sufficiently reliable to distinguish between the time of transit as originally stated and that given by the formula.) This value would be insignificant in flood routing over a reach involving flood-wave travel time of 15 hours.

The writer does not claim that the upgrading of flood crests by storage action has been proved, but merely offers the explanation and the examples of the paper in support of the statement, "There is some valley storage which does not decrease flood peaks." The upgrading of hydrographs does appear possible, however, and should be investigated. A closely analogous situation exists in electrical circuits, and the upgrading (in height, not discharge) of hydraulic waves in converging channels is a recognized phenomenon in tidal hydraulics.

Mr. Linsley clarifies several ideas in the paper by his discussion. The formula he presents for determining time of concentration in watersheds without records shows better correlation with data in the paper than the author's formula. To aid in further study by others, additional data are presented in the same manner in Table 6. The majority of values were developed from records available prior to 1938. The variation in the constants suggests judicious local application. Attention is again directed to the first two sentences under the heading, "Determination of the Instantaneous Hydrograph for Streams Without Flow Records," and to Mr. Snyder's closing two paragraphs.

Messrs. Linsley and Johnstone properly question the writer's use of uniform velocity in establishing time contours along the watercourses and suggest variations based on slope and efficiency of watercourses. These refinements were considered, but investigation showed their use to be unaccompanied by superior results. This may be a case of two wrongs making a right, as the likewise neglected principle of using greater storage values,  $K$ , on flows from more remote areas instead of a single average value is a logical refinement of opposite effect. In general, the use of a varied velocity of channel flow and a single value of  $K$  tends to increase the fault originally recognized by the writer in item 6(b) of the "Summary."

Messrs. Brater and Johnstone voice an interest in a more detailed description of the time-area concentration curve. As Mr. Snyder states, the curve is an old device<sup>42,43</sup> for calculating flow. It has fallen into disuse because its theory assumes the reductive influence of storage to be negligible. A storage correction factor was included in the paper to revitalize this useful tool. The time-area concentration curve is derived by marking time contours along the watercourses, the time to the most remote point being equal to the "time of

<sup>42</sup>"Sewerage and Sewage Disposal," by L. Metcalf and H. P. Eddy, McGraw-Hill Book Co., Inc., New York, N. Y., 1922, p. 94.

<sup>43</sup>"Calculation of Flood Discharge by the Use of a Time-Contour Plan," by Cecil N. Ross, *Transactions, Institution of Engrs. (Australia)*, Vol. II, 1921, p. 85.

concentration." For the latter term, a standard definition such as that quoted by Professor Johnstone is intended. Fig. 4 illustrates why the terminus of this time is not at the end of the hydrograph (theoretically infinite in length) but well up on the recessional leg.

TABLE 6.—ADDITIONAL COMPARISON BETWEEN EQS. 18a AND 18b

Stream	s (ft per mile)	L (miles)	K (hours) <sup>a</sup>	A (sq miles)	C <sub>1</sub> (Eq. 18a)	C <sub>2</sub> (Eq. 18b)
North Branch, Potomac River, at Bloomington, Md.	45	49	10	287	1.4	0.081
Savage River at Bloomington, Md.	65	26	11	115	3.4	0.32
Georges Creek at Franklin, Md.	78	18	5	72	2.4	0.29
Wills Creek near Cumberland, Md.	40	35	8	247	1.4	0.092
Patterson Creek near Headsville, W. Va.	19	40	12	216	1.3	0.099
South Branch, Potomac River, near Petersburg, W. Va.	24	60	15	642	1.2	0.018
South Fork, South Branch, Potomac River, near Moorefield, W. Va.	25	54	13	283	1.2	0.071
Cacapon River at Yellow Spring, W. Va.	16	51	10	306	0.78	0.044
Back Creek near Jones Springs, W. Va.	11	40	10	243	0.81	0.052
Conococheague Creek at Fairview, Md.	9.8	62	30	494	1.5	0.068
North River near Burkettown, Va.	37	18	9	375	3.0	0.16
Middle River near Grottoes, Va.	14	54	12	360	0.53	0.044
South River at Waynesboro, Va.	17	24	12	144	2.1	0.17
North Fork, Shenandoah River, near Strasburg, Va.	12	95	20	772	0.72	0.026
Passage Creek at Buckton, Va.	31	32	6	87	1.0	0.11
Cedar Creek near Winchester, Va.	30	26	6	101	1.3	0.12
Antietam Creek near Sharpsburg, Md.	11	49	24	281	1.6	0.097
Monocacy River near Frederick, Md.	5.9	62	28	817	1.1	0.038
Goose Creek near Leesboro, Va.	11	33	17	338	1.7	0.093
Seneca Creek at Dawsonville, Md.	17	17	6	101	1.5	0.15
Jackson River at Falling Spring, Va.	22	47	12	409	1.2	0.059
James River at Lick Run, Va.	17	78	28	1,369	1.5	0.040
Cowpasture River near Clifton Forge, Va.	13	69	24	456	1.3	0.059
Calpasture River at Goshen, Va.	23	36	12	147	1.6	0.13
North River at Rockbridge Baths, Va.	27	45	18	329	2.1	0.12
North River near Lexington, Va.	21	58	20	487	1.6	0.071
North River near Buena Vista, Va.	16	70	24	649	1.3	0.053
Hardware River near Scottsville, Va.	17	22	18	116	3.4	0.32
Slate River near Arvonnia, Va.	9.0	34	30	235	2.6	0.17
Rivanna River at Palmyra, Va.	8.0	50	32	675	1.8	0.070
Willis River at Flanagan Mills, Va.	4.2	55	72	247	2.7	0.17
Appomattox River at Farmville, Va.	8.7	34	32	306	2.8	0.16
Appomattox River at Mattoax, Va.	3.0	87	114	729	2.3	0.084
Appomattox River near Petersburg, Va.	2.9	122	168	1,335	2.3	0.064
Meherin River near Lawrenceville, Va.	3.2	67	60	553	1.6	0.068
Meherin River at Emporia, Va.	2.4	86	78	750	1.4	0.051
Roanoke River at Roanoke, Va.	17	48	12	388	1.0	0.052
Blackwater River near Union Hall, Va.	14	46	18	208	1.4	0.10
Pig River near Totes, Va.	8.7	72	24	394	0.98	0.049
Snow Creek at Sago, Va.	18	18	11	60	3.2	0.41
Goose Creek near Huddleston, Va.	14	33	12	187	1.4	0.099
Otter River near Evington, Va.	18	34	22	325	2.7	0.15
Falling River near Brookneal, Va.	11	23	22	228	3.2	0.22
Dan River near Francisco, N. C.	35	43	4	124	0.55	0.019
Mayo River near Price, N. C.	20	38	14	260	1.6	0.10
Smith River at Bassetts, Va.	17	37	7	265	0.77	0.047
Smith River at Martinsville, Va.	13	48	13	374	0.98	0.051
Smith River at Spray, N. C.	9.4	71	23	538	0.99	0.043
Sandy River near Danville, Va.	21	21	5	113	1.1	0.10
Banister River at Halifax, Va.	7.2	46	60	552	3.5	0.15

<sup>a</sup> Time of concentration, not reductive storage factor.

Mr. Crawford presents interesting analyses of channel-storage correlation with stages at the discharge point and upstream from it, and shows the superiority of correlations which give weight to upstream values of inflow and stage. He calls attention to the potential possibilities of determining from discharge records other storage relations which would be too costly by cross-sectional surveys. The writer has seen area-capacity curves for small dam sites so developed which would require very detailed field surveys for equal accuracy.

Mr. Kennedy emphasizes the fallacy of constant base length in a unit hydrograph. Much of this fallacy may originate in the assumption of a finite base length. If storage theories and unit hydrograph theories are to be correlated, the concept of finite base length in a unit hydrograph must give way to a concept of an infinite length, a characteristic possessed by the writer's instantaneous hydrographs and their derivatives. Nevertheless, the practical differences, as shown by Fig. 15, are within the range of usually acceptable hydrograph computations. This is probably the reason that the practical simplicity of finite constant base length has so long outweighed the theoretical superiority of infinite base length.

Mr. Cochrane has very thoroughly examined the procedure for deriving unitgraphs utilizing storage concepts, and presents some explanations which should be helpful to those who were mystified by the writer's presentation. In a simple manner, he clarifies the twofold effect of storage: First, the greater effect of storage in creating time lags between flows from different zones, or the time-area diagram; and, second, the further modification of hydrograph shape accomplished by storage.

The order in which these steps have been utilized is the reverse of that in Fig. 8, but the result is the same regardless of order. The order in Fig. 16 gives a clear picture of the mechanical effects, and would be necessary if one wished to use variable values of  $K$  for different zones. At the sacrifice of this possibility, the order of Fig. 8 involves less work. However, a unit hydrograph for an entire area may produce very disappointing results if applied to a storm characterized by great nonuniformity of runoff distribution. Mr. Cochrane presents a very practical application of storage principles which considers both nonuniformity of runoff distribution and some variation in storage effect.

Another method of accounting for nonuniformity, suggested by Mr. Turner, is to extend the time-area concentration curve to express not only shape but also different runoff distribution—that is, multiply the zone areas by the depth of runoff in each zone before embarking on the routing procedure of Table 1. This procedure takes care of distribution with respect to area and is a fine tool for studying the magnitude of the effect of areal nonuniformity of runoff distribution.

Both procedures abandon the unit hydrograph and require complete recalculation of each flood with attendant increased labor. Other less logical, but extremely useful techniques to account rapidly for nonuniformity include adjustments of chronological sequences of runoff used with the unit hydrograph. Thus, concentration of volume on more remote parts of a simple watershed can be simulated in some degree by using an effective runoff volume for a later time than actually was the case. Extreme intensity of rainfall within a unit period can be simulated by increasing the computational amount in that period and decreasing the values on either side of the period. The most flexible technique yet utilized by the writer is the development of unit hydrographs for very small zones, which are routed downstream to gaging stations and tabulated in the manner of Table 2. This procedure is followed for as small areas as desired to express nonuniformity to the necessary degree. These sub-unitgraphs can always be added together for use on areas as large as justified

by the uniformity of any storm and can be used separately if necessary. Although this procedure was discussed under the heading, "Unit Hydrographs for Large Drainage Areas," any watershed is large from the viewpoint of detailed, exacting hydrograph computation.

Refined techniques are necessary and desirable to establish more suitably correct procedures and to confirm academic viewpoints as to the effect of certain qualitative variables. However, a computation is only a guide to judgment. After a hydrograph is computed with a unitgraph or other appropriate tool, judgment demands some modification of the answer if the premises of the methods and the facts of the occurrence are not in accord. At times a wide line, a small scale, or a little freehand "artistic license" may have as much merit as hours of computing. In their proper place, these too are proper tools.

Mr. Snyder presents several of the practical aspects of hydrograph calculation and the mathematical basis for a very simple and useful method of routing over long reaches of river where storage-discharge relationships are such as to produce essentially constant time elements. He calls attention to elements of logical and mathematical fallacy which may appear small to some investigators, but which must be faced squarely and solved before the hydrograph computations can be considered accurate enough to prove or disprove some of the questions raised in this paper.

As Mr. Snyder states, the Muskingum method of flood routing loses its usefulness when applied to too long reaches. However, this may also occur because the storage-discharge relationship for a long reach of river is much less closely approximated by a straight line than is the relationship for short reaches. The storage item in flood-routing formulas such as Eq. 5 deals with the volume stored in a reach at any instant of time, whereas the storage capacity of channels having nearly constant elements of flow time ( $n = 1.0$  in Eq. 3) refers more closely to the volume under the high-water profile. Although this latter volume is the total storage volume utilized in the reach of river during flood passage, it is more than is used at any instant during that time. The difference becomes a material one in long reaches. Nevertheless, if storage solutions are confined to the length of reach for which Eq. 5 would be applicable, Eq. 6b is a correct solution. The Muskingum method is a correct solution of Eq. 6b only to the extent that a finite value of  $T$  is a satisfactory approximation of  $dt$ . Mr. Snyder expresses the opinion that values of  $T$  used in the preparation of Figs. 2 and 3 were too short.

The inadequacies of methods of mathematical solution receive too little recognition. Most engineers find it necessary to adopt arithmetic approximations, such as Eqs. 1 and 6c, for the solution of calculus equations, such as Eqs. 2a and 6b.<sup>44</sup> The practises are so common that they escape much comment. The simplicity of the Muskingum method encourages these procedures, since it depends first on the substitution of a finite value of time  $T$  for the infinitesimal time  $dt$ . Then, too, many convenient, quick solutions evolve from various lengths of time  $T$  which make one or more of the constants either 1.0 or 0. Values of  $T$  as large as  $K$  or larger conveniently obscure some of the

<sup>44</sup> "Graphical Integration of the Flood-Wave Equations," by Harold A. Thomas, *Transactions, Am. Geophysical Union*, Vol. 21, 1940, Pt. I, p. 597.

effects of the inadequacy of the theory recognized by the writer in his discussion of Fig. 2, and have been accepted by some as a basic requirement. The differences are as given by Mr. Snyder. In an effort to prove whether such a basic requirement added anything to the over-all accuracy of hydrograph solutions, the writer has for several years assumed that the shortest practical time element,  $T$ , was the most appropriate. In general, practical applications have been no more faulty, and in some cases apparently better, whereas the practical benefits of greater mathematical consistency and rapid checking procedures accrue to (or result from) closer adherence to the mathematical premises.

Mr. Snyder's presentation of the mathematics of a simple method of lag routing should increase the practical appeal and use of this simple procedure. To be completely general, the solution may need some provision (1) to cover the flow conditions in which the lag of the crest discharge is greater than the lag between the centers of mass of the inflow and outflow, and (2) to cover the conditions in which the recession rate of outflow exceeds the recession rate of inflow. Flow conditions requiring this modification are evident in Fig. 18, in which the hydrographs of the Appomattox River at Mattoax are shown to have a materially smaller rate of recession than does that portion of the hydrograph at Petersburg which encompasses the flow volume from above Mattoax.

Inspection of Fig. 17 and Eq. 25 suggests the applicability of a negative value of  $T_s$ . The writer finds the mathematics of a solution evasive, however, and doubts that this is a practical solution. Nevertheless, it is explanatory of a relationship between  $K$  and time of concentration ( $T_c$ ), which still baffles him. However, it is assumed for the purpose of the next paragraph that negative values of  $T_s$  are applicable.

The failure of the writer to solve this possible adaptation of lag routing is the source of his confusion in statements under the heading, "The Hydrograph as an Index of Storage," about which Mr. Snyder rightfully comments. The values of  $K$  applicable to smaller watersheds appear to be as large as 0.5 of the time of concentration ( $T_c$ ); yet those for larger watersheds have a much smaller ratio of  $T_c$  and occasionally a smaller dimensional quantity. Obviously this is possible if  $T_s$  could have a negative value. For the three gages on the Appomattox River—Farmville, Mattoax, and Petersburg, each site being downstream from the preceding site—the values of  $K$  which the writer regards as most appropriate are, respectively, 9 hours, 12 hours, and 9 hours, whereas the most acceptable times of concentration,  $T_c$ , are 32 hours, 114 hours, and 168 hours. (These values for Petersburg differ from values in Table 1, which were the very first computations made for that stream. The difference between the values of  $T_c$  equal to 144 hours and 168 hours is dimensionally large, but the same percentage difference in the more usual range of the writer's application, 24 hours or less, would represent but 4 hours, which is close to the limiting error fixed by rainfall data. Furthermore, that percentage difference is quite comparable with the expected difference due to seasonal variation in flow conditions, a factor usually ignored, but by no means absent.)

The Appomattox River presents the most perplexing flow conditions to which the writer has yet applied the unit hydrograph theory and the longest

time of concentration for comparable drainage area that he has ever encountered, as well as a unique shape. Although the illustrated hydrographs for the Smith River at Bassetts and the James River at Lick Run are certainly the more common type, exceptions, like the illustrated hydrographs of Appomattox River and Meherrin River, are necessary to prove the rule. Judging from records of rounded and tree-shaped watersheds typical of the glaciated and alluvial streams of the Midwest, it would appear illogical that a stream could rise for a longer period than it fell, or that streams could have unit hydrographs like those presented for the Appomattox and Meherrin rivers. Mr. Sherman's doubt that any unit hydrographs were presented in the paper is therefore very understandable.

Mr. Turner provides a well-balanced discussion of the paper, confirming statements and expressing appropriate reserve and doubt where his experience and understanding of conventional concepts and procedures do not justify presented conclusions. His comparison of results obtained from the outlined unitgraph procedure and from his similar procedure<sup>31</sup> encourages confidence in the principles applied. His extension of the method to include varying rates of rainfall on different parts of the area is quite practical.

Professor Johnstone outlines some logical elements of greater and lesser refinement whose merit would depend on the extent of the use for them. He expresses several views in sharp conflict with those of the paper. The propriety of defining a unitgraph to include nonsurface runoff is basically criticized. Granting that some investigators have presented hydrographs attributed to surface runoff, the writer questions whether such a claim can be substantiated or disproved, since there is no feasible method of distinction, no acceptable specific definition of either kind of runoff, or any impervious boundary between the flow channels utilized by each. Practise in the application of the terms seems to justify distinctions satisfactory to the user and determinable only to the extent of their compatibility with a preferred theory. The writer indicated that separate unitgraphs could be used for surface and subsurface flow, if warranted, and that there was nothing fixed about the 70:30 ratio. However, in computations of subsurface flow there seems little reason to reject that definable quality which the use of the unit hydrograph theory has provided in surface flow analysis.

In his demonstration that there " \* \* \* should \* \* \* be no possibility of multiple solutions" of conventional unit hydrograph derivations from rainfall and stream flow data only, Professor Johnstone adequately presents the reasons why an investigator who uses all the data available can secure: (a) A distressingly large number of different solutions if he assumes he is able to determine runoff from rainfall; or (b) no solution if he admits inability to determine runoff from rainfall. In the storage concepts presented in the paper, there are additional considerations which eliminate the necessity for dependence on rainfall and runoff determinations. Until runoff determinations become more reliable, however, the real accuracy of the method must remain partly shrouded in unproved hope.

<sup>31</sup> "The Flood Hydrograph," by Howard M. Turner and Allen J. Burdoin, *Journal*, Boston Soc. of Civ. Engrs., Vol. XXVIII, No. 3, July, 1941, p. 232.

# AMERICAN SOCIETY OF CIVIL ENGINEERS

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## DISCUSSIONS

### TRANSLATORY WAVES IN NATURAL CHANNELS

#### Discussion

BY M. A. CHURCHILL

M. A. CHURCHILL,<sup>14</sup> Assoc. M. Am. Soc. C. E.<sup>14a</sup>—The Society is indebted to the author for the presentation and interpretation of a wealth of observed, not theoretical, data on translatory waves in natural channels. In an attempt to make the paper of still greater value, the writer presents herewith some sup-

TABLE 11.—SPEED OF TRAVEL AS OBSERVED, IN 1943, ON THE LOWER HOLSTON RIVER IN TENNESSEE

STATION		Distance, in miles	(a) WAVE TRAVEL, FALLING FACE; Nov. 7			(b) WAVE TRAVEL, RISING FACE; Nov. 8			(c) WATER TRAVEL; Nov. 12 AND 13 <sup>a</sup>	
Location	Mile		Clock time at midpoint of fall or rise	Time in hours	Ob- served rate <sup>b</sup>	Clock time at midpoint of fall or rise	Time in hours	Ob- served rate <sup>b</sup>	Time in hours	Ob- served rate <sup>b</sup>
Below Cherokee Dam	52.0	18.8	12:10 a.m.	4.50	4.17	12:01 a.m.	5.33	3.52	8.25	2.28
Nances Ferry	33.2	16.2	4:40 a.m.	4.67	3.47	5:20 a.m.	3.83	4.22	9.00	1.80
Straw Plains	17.0	11.6	9:20 a.m.	2.50	4.63	9:10 a.m.	3.00	3.87	5.50	2.11
Highway 70 Bridge	5.4	46.6	11:50 a.m.	11.67	(4.00) <sup>c</sup>	12:10 p.m.	12.16	(3.83) <sup>d</sup>	22.75	(2.05)

<sup>a</sup> Average flow was 7,900 cu ft per sec. <sup>b</sup> In miles per hour. <sup>c</sup> Theoretical wave travel rate, 3.88 miles per hr. <sup>d</sup> Theoretical wave travel rate, 3.95 miles per hr.

porting data on observed waves in natural channels and in reservoirs, together with corresponding data on observed rates of water travel.

NOTE.—This paper by J. H. Wilkinson, M. Am. Soc. C. E., was published in June, 1944, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: January, 1945, by Harold A. Weggel, and W. M. Lansford.

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<sup>14a</sup> Received by the Secretary December 28, 1944.

**Wave Travel in Lower Holston River.**—In connection with a determination of the time required for water to travel 46.6 miles from Cherokee Dam on the

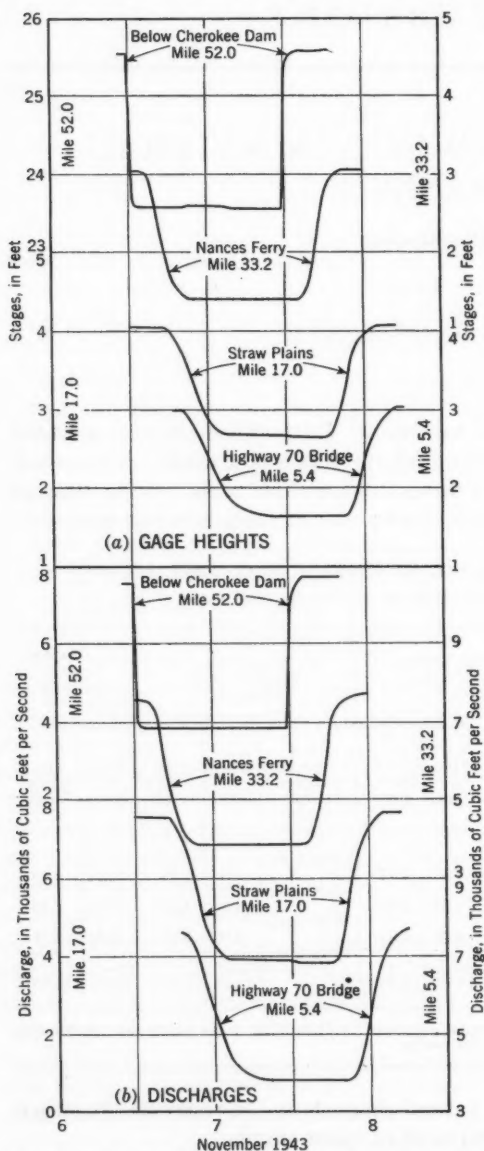


FIG. 10.—STATION OBSERVATIONS ON THE LOWER HOLSTON RIVER, TENNESSEE, NOVEMBER 7 AND 8, 1943

Holston River to a point near the head of Fort Loudoun Reservoir above Knoxville, Tenn., the velocity of a negative wave of translation was also observed. The negative wave was produced by reduction for a period of 24 hr in the discharge of Cherokee Dam on November 7, 1943, from 7,530 cu ft per sec to 3,840 cu ft per sec, and then an abrupt increase to 7,740 cu ft per sec. This negative wave was observed at four stations on the Holston River below Cherokee Dam as shown by Fig. 10(a) (stages) and Fig. 10(b) (discharges). Stages were observed at all four stations, but discharges are known only at the station immediately below Cherokee Dam and at Straw Plains. (Stages below Cherokee Dam and at Straw Plains are from continuous recorder charts. Stages at Nances Ferry and at Highway 70 Bridge are from staff gage readings at intervals of 2 hr or less.) Discharges at Nances Ferry and at Highway 70 Bridge were approximated. Table 11 gives pertinent data on this negative wave. The theoretical rates of travel (footnotes (c) and (d), Table 11) were computed by Eq. 1b, based on discharges and areas at the two gaging stations in the reach. It is recognized that the relatively good agreement between observed and

computed wave velocities, when the computed velocities are based on only two, relatively short, rated river reaches, is "more luck than sense." The

author necessarily based his comparisons of observed and computed rates on similar data and so it is rather remarkable that his results on Clinch River show such good agreement with the theory.

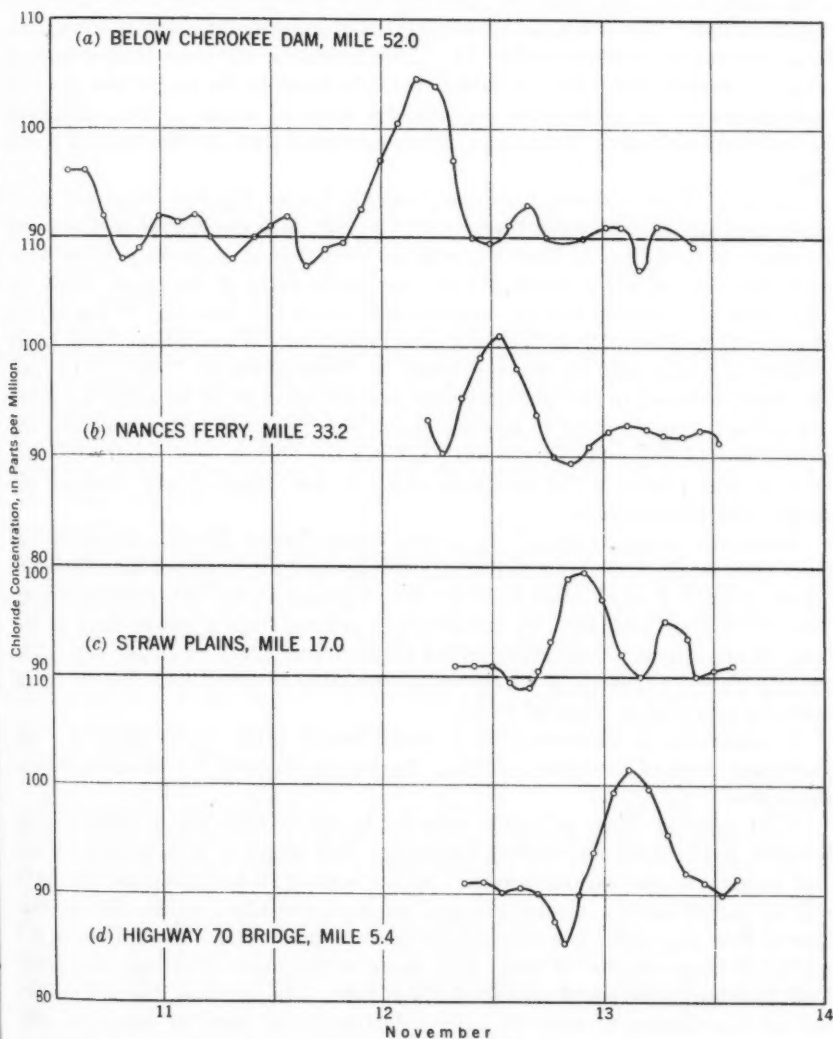


FIG. 11.—CHLORIDE CONCENTRATIONS AT OBSERVATION STATIONS ON THE LOWER HOLSTON RIVER, NOVEMBER 12 AND 13, 1943

*Water Travel in Lower Holston River.*—The rate of water travel, as distinguished from wave travel, through the same reach of river was observed between November 12 and 14, 1943, during which time the release from Cherokee Dam was nearly the same as both before and after the negative wave

generated on November 7, 1943. (The release from Cherokee Dam on November 12 increased gradually from 7,770 cu ft per sec at 1:00 a.m. to 8,070 cu ft per sec at midnight.) To determine the rate of water travel, samples were collected at 2-hr intervals from the same four points at which the wave was observed. These samples were analyzed chemically for chloride concentration, with results as shown in Fig. 11. The presence in the river of an industrial waste of varying chloride concentration made possible the use of the chloride concentrations for identifying a particular mass of water at the successive downstream stations. Table 11(c) gives pertinent data on the rate of water travel.

*Ratio of Wave Velocity to Water Velocity, Lower Holston River.*—A comparison of the rate of wave travel with the rate of water travel in the lower Holston River indicates that the rate of wave travel is slightly more than twice the rate of water travel, for the particular rates of discharge observed. This ratio tends to support the author's contention (see heading, "Translatory Waves on the Clinch River: Velocity of a Translatory Wave") that "The water velocity is about half the wave velocity at ordinary stages \* \* \*." In fact, the data presented in this discussion on the relation of wave to water velocities are probably better proof of this statement (as it applies to the general case) than the data presented by the author, since the author used water velocities at only four points in the 64.2-mile reach of the Clinch River between the Norris and Wheat gages.

*Wave and Water Travel, Watauga and South Holston Rivers.*—Observations have been made of the time required for water and for waves of translation to travel through a reach of 40 miles on the Watauga and South Holston rivers. The waves were produced by variations in release from a power dam at the head of the reach. Discharges varied from 200 to 1,300 cu ft per sec. The average time for a series of waves was determined to be 16 hr. Water travel over the same reach required 91 hr.

Comparison of wave-travel and water-travel rates shows that, for the particular rates of discharge existing, the waves traveled 5.7 times as fast as the water.

This ratio of wave to water velocity is rather high for a natural river channel, particularly considering the rather wide range of flow between crests and troughs of the various waves. As the author states, however, the ratio will be higher for low discharges, and during these observations the average rate of flow was about one third of the mean annual rate. The existence of a relatively large number of long, deep pools in the lower Watauga and South Fork Holston rivers could explain the high ratio. In any case, the wide difference in the relation of wave to water velocities found here, as compared with the relation found on the lower Holston River, indicates the possibilities of error in trying to use wave velocities as a basis for computing water velocities without rather complete data on physical conditions of the reaches involved.

*Wave and Water Travel, Fort Loudoun Reservoir.*—Observations of wave travel through Fort Loudoun Reservoir (42.8 miles) showed an over-all wave velocity of 21.4 miles per hr. The theoretical rate of wave travel was computed by Eq. 2b. The mean depth of the reservoir was found to vary essen-

tially as a straight line from 14 ft at the upper gage to 48 ft at the dam, when the pool elevation is 811.5. The wave velocity, except for the influence of the water velocity, can be expressed as  $\sqrt{g y_m} = \frac{dL}{dt}$ ; or:

$$dt = \frac{dL}{\sqrt{g y_m}} \dots \dots \dots (7)$$

Therefore,

$$t = \int_0^L \frac{dL}{\sqrt{g y_m}} = 0.176 \int_0^{226,000} \frac{dL}{\sqrt{14 + 0.0001504 L}} \dots \dots \dots (8)$$

$$\text{and } t = 0.176 \left( \frac{2\sqrt{14 + 0.0001504 L}}{0.0001504} \right)_0^{226,000} = 7,460 \text{ sec} = 2.08 \text{ hr.}$$

A total time of 2.08 hr represents an average velocity through the 42.8-mile ( $L = 226,000$  ft) reach of reservoir of 20.6 miles per hr. Due to thermal stratification and a density underflow, water velocities on the surface existed only in the upper end of the reservoir. Therefore,  $V_1$  in Eq. 2b was essentially zero. The theoretical wave velocity is thus computed to be 20.6 miles per hr as compared with the observed velocity of 21.4 miles per hr.

Measurements of actual water travel by chemical identification methods indicate that a period of 190 hr is required through the 42.8 miles of reservoir. This rate of travel is 0.22 mile per hr. Had the moving water occupied the full depth of the reservoir throughout the entire length of the pool, the water velocity would have been even slower than the observed value.

The wave velocity was 95 times as fast as the water velocity, over-all. This ratio explains why the author's conclusion is true that the water velocity has little influence on the wave velocity in a reservoir.

*Basic Concepts of the Relation of Wave to Water Velocity.*—In an attempt to get a clear understanding of exactly how a translatory wave could run ahead of the water which originally formed it, the writer formulated the following relations several years ago:

Assume a smooth, straight natural channel in which the flow is gradually increased from a steady flow of  $Q_1$  to a higher flow  $Q_2$ . Under such conditions, a profile of the channel and water surface would appear as shown in Fig. 12. Since the hydraulic radius is greater for the higher flow  $Q_2$ , the velocity  $V_2$  is greater than  $V_1$ , assuming that  $Q_2$  is less than bankfull discharge. Since this is the case, water moving at mean velocity  $V_2$  tends to overtake water moving at mean velocity  $V_1$ . This action produces a result similar to an express train crashing into the rear end of a slow freight. In that case the rear cars of the freight are telescoped by the express. In the river, the slower moving water is "telescoped" by that moving at velocity  $V_2$ . The quantity of water per second so "telescoped" is expressed mathematically by  $(V_2 - V_1) A_1$ , in which  $A_1$  is the cross-sectional area corresponding to  $V_1$ . This quantity of water is actually squeezed out of its original position in the river channel and is forced into that

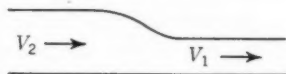


FIG. 12.—TRANSLATORY WAVE IN A CHANNEL OF UNIFORM WIDTH

portion of the channel represented by  $A_2 - A_1$ . This quantity of water is thereby added each second in front of that advancing at the velocity  $V_2$ . Therefore, the rate of advance of the wave front is  $\frac{(V_2 - V_1) A_1}{A_2 - A_1}$ , relative to  $V_2$ .

The actual rate of advance of the face of the wave is  $V_2 + \frac{(V_2 - V_1) A_1}{A_2 - A_1}$ .

From this expression it is obvious that the wave velocity is always greater than the water velocity, provided the channel has an initial flow in it, since the second term is always positive. By simple algebra, this expression for the wave velocity can be shown to be equal to the familiar expression, Eq. 1b, except for the constant used to convert feet per second to miles per hour.

*Wave-Height Reduction.*—The author explains what happens to a translatable wave produced by a release of short duration, when its height is reduced (heading, "Characteristics of Translatory Waves"), without explaining the reasons for these happenings. The author states that " \* \* \* the spreading fillets at the end-of-rise and at the beginning-of-fall gradually shorten this [crest] period as the wave progresses downstream." It seems to the writer that, in addition to any so-called "channel-storage" effects, the basic reasons why the crest of a wave of short duration is gradually lowered are as follows: As the wave moves downstream, it pushes the water already in the channel ahead of it, as explained previously herein, and forces this water into that area of the channel represented by  $A_2 - A_1$ . However, this pushed water is not spread uniformly over the area  $A_2 - A_1$ , but rather it tends to resist movement and stay near the lower edge of this area, where the channel width is the narrowest. This nonuniformity tends to push the beginning-of-rise point out farther and farther ahead of the crest of the rising wave. As the nose runs farther ahead, the rising face of the wave is thereby flattened; only a little water is added to the rising face of the wave, at and near the crest.

Although little water is added on to the forward end of the crest, a considerable quantity is removed from the rear. Since water at the crest stage moves at a velocity higher than that at the lower stage to the rear, a quantity of water equal to  $(V_2 - V_3) A_3$  is removed from the rear face of the wave each second. This constant drain on the rear face of the wave eats it away and thus shortens the crest length until finally the original crest is all gone. Then the remaining crest is still further flattened until finally no wave exists.

An additional reducing influence is the fact that, due to the increased slope of the rising face, the water composing it tends to run faster than the falling face where the slope is flatter. Thus the wave tends to stretch out in length at the base, and consequently is lowered at the crest.

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## DISCUSSIONS

### AMPLIFIED SLOPE DEFLECTION

#### Discussion

BY JOHN E. GOLDBERG, AND CAMILLO WEISS

JOHN E. GOLDBERG,<sup>9</sup> ASSOC. M. AM. SOC. C. E.<sup>10</sup>—An amplification of the slope-deflection method is presented in this concise and excellent paper. The fundamental geometry of slope deflection is combined with the author's concept of the traverse angle as applied to members under flexure. The writer is gratified to perceive that, as this and other interesting papers indicate, the value of slope deflection as a powerful method of analysis is steadily becoming more appreciated.

Several of Mr. Stewart's statements deserve critical consideration. One of these is the sentence (see heading, "Demonstration Problems"): "In conventional slope deflection the number of simultaneous equations required to solve a given problem is definite." This standardization is a distinct advantage of the slope-deflection method from which both the experienced and the inexperienced engineer will benefit since any engineer can see at once the proper direct slope-deflection approach to a given problem. On the other hand, the slope-deflection method is so extremely flexible that it allows much leeway, when desired, in the solution of special problems.

Consider, for example, the problem of Fig. 1. Some years ago the writer developed an extremely simple analysis for problems of this type.<sup>10,11</sup> This method has the advantages of being readily applicable to any number of stories and of being extremely rapid since the basic equations converge very quickly even in extreme cases. Applying the method to Mr. Stewart's first problem, two equations would be set up immediately:

$$\theta_A (6 \times 2 + 1) = 0.5 (20 \times 10) + \theta_D \dots \dots \dots (19a)$$

and

$$\theta_D (6 \times 2 + 1) = 0.5 (20 \times 10) + 0.5 (20 \times 20) + \theta_A \dots \dots (19b)$$

NOTE.—This paper by Ralph W. Stewart, M. Am. Soc. C. E., was published in September, 1944, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: November, 1944, by Leon Benkin.

<sup>9</sup> Engr., Consolidated Vultee Aircraft Corp., San Diego, Calif.

<sup>10</sup> Received by the Secretary November 30, 1944.

<sup>11</sup> *Transactions*, Am. Soc. C. E., Vol. 102 (1937), p. 922.

<sup>12</sup> *Proceedings*, Am. Soc. C. E., May, 1944, p. 723.

The solution of Eqs. 19 ( $\theta_A = 9.524$  and  $\theta_D = 23.81$ ) agrees exactly with the author's solution. Thus, the method is shown to be simple and direct. Of course, this is merely proof of the fact that the most useful method generally is the one with which the engineer is most familiar.

Mr. Stewart goes on to mention the same frame with the lower girder pinned at one end and states that six simultaneous equations are required by the conventional slope-deflection method. This is absolutely true; but, when these equations are set up (and setting them up is in itself very simple), three of the equations may be eliminated in a very direct manner. Note also that a conventional strain-energy analysis can be made in terms of three redundants.

Mr. Stewart's point concerning sign conventions (see heading, "Definitions of Symbols and Other Data: Signs," and text following Eqs. 7) is well taken. In his original presentation<sup>12</sup> of slope deflection, G. A. Maney, M. Am. Soc. C. E., advanced the sign convention which the author advocates. In subsequent work,<sup>13,14,15,16</sup> both Professor Maney and the writer continued to use that convention, finding it simple and convenient.

The author's statement, discussing Eqs. 7, that " \* \* the final moment at the end of a beam for which the moments are not dominated by joint movements is less than the fixed-end moment" is likely to be misinterpreted by the reader. The statement would be clarified greatly if the author had written that "the moment is less, algebraically, than the fixed-end moment."

CAMILLO WEISS,<sup>17</sup> M. Am. Soc. C. E.<sup>17a</sup>—The four "properties of an elastic curve traverse" presented in this paper are principles derived in part from "moment areas" and are important enough to be compared with the well-known principles of the moment-area method.

The concept of the  $\Delta$ -angle is helpful in visualizing the effect of variations in cross sections or end restraints. The author is justified in stating that in many cases the number of simultaneous equations may be reduced by recognizing the significance of the  $\Delta$ -angles.

The method would seem to deserve a more descriptive designation than "amplified slope deflection." Although related to slope deflection, the unknowns of the author's method consist exclusively of slopes and the method is based on its own distinctive principles.

The author should be commended for his clear and able presentation of these principles and for his skilful derivation and interesting interpretation of the slope-deflection equations.

<sup>12</sup> *Engineering Studies No. 1*, by G. A. Maney, Univ. of Minnesota, Minneapolis, 1915.

<sup>13</sup> "Vertical-Load Analysis of Rigid Building Frames Made Practicable," by John E. Goldberg, *Engineering News-Record*, November 12, 1931, pp. 770-772.

<sup>14</sup> "Simplified Methods for the Analysis of Multiple Joint Rigid Frames," by George A. Maney and John E. Goldberg, *Northwestern University Bulletin*, Vol. XXXIII, No. 7, 1932.

<sup>15</sup> "Wind Stresses by Slope Deflection and Converging Approximations," by John E. Goldberg, *Transactions, Am. Soc. C. E.*, Vol. 99 (1934), pp. 962-985.

<sup>16</sup> "Natural Period of Vibration of Building Frames," by John E. Goldberg, *Journal of the American Concrete Institute*, September, 1939, p. 81.

<sup>17</sup> Designer, Bethlehem Steel Co., Fabricated Steel Constr., Eng. Dept., Bethlehem, Pa.

<sup>17a</sup> Received by the Secretary January 16, 1945.

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## DISCUSSIONS

### ANALYSIS OF STATICALLY INDETERMINATE STRUCTURES USING REDUCED EQUATIONS

#### Discussion

BY JOSEF SORKIN

JOSEF SORKIN,<sup>5</sup> M. AM. SOC. C. E.<sup>5a</sup>—The thesis presented in this paper is offered (see "Synopsis") as one that " \* \* \* is useful in simplifying the analysis of certain types of statically indeterminate structures \* \* \*." The paper belongs to the category of a number of others published since 1930, all attempting to simplify the application of the classical elastic theory and nearly all, unfortunately, ending in utter failure. The only notable exception is the "Analysis of Continuous Frames by Distributing Fixed-End Moments,"<sup>6</sup> by Hardy Cross, M. Am. Soc. C. E., in which a radically new approach substitutes simple arithmetic for the solution of simultaneous equations. This statement implies no reflection upon the sincerity of purpose of the other writers.

The simple fact is that most statically indeterminate structures, as encountered in actual practice, do not lend themselves to solution by specific simple mathematical expressions. Various solutions may be devised for any assumed academic cases, but it is impossible to assume enough cases to meet all, or even most, of the conditions encountered in actual design problems. That is primarily the reason why efforts to develop a "cure-all formula" are futile.

The author states clearly that the method proposed does not replace other methods and that it does not apply to all structures. Its primary purpose appears to be simplification of approximations of members preliminary to final design. The design of a statically indeterminate structure by any method requires that the size and the properties of the members must be assumed before proceeding with the stress analysis. It is not clear why the same principle does not apply to the method proposed by the author, regardless of his statement to the contrary. Irrespective of the method used for determining the reactions, the shape of the member originally assumed must finally be checked for stresses.

NOTE.—This paper by Lee H. Johnson, Jr., was published in November, 1944, *Proceedings*.

<sup>5</sup> Engr. of Design, Howard, Needles, Tammen & Bergendoff, Kansas City, Mo.

<sup>5a</sup> Received by the Secretary January 2, 1945.

<sup>6</sup> "Analysis of Continuous Frames by Distributing Fixed-End Moments," by Hardy Cross, *Transactions, Am. Soc. C. E.*, Vol. 96 (1932), p. 1.

Any part of the structure found to be either deficient or extravagant must be modified correspondingly with suitable readjustment of the reactions and the stresses. In prognosticating the proportions of such a structure, nothing can ever take the place of a designer's judgment based on past experience with similar structures. The question appears to be whether the methods proposed are in fact simpler than other methods available.

In so far as two-hinged arches are concerned, certain rudimentary rules have been in existence for many years. Thus, for the preliminary design of a parabolic arch, it is customary to assume that<sup>7</sup>

$$H = \frac{5}{8} \frac{P L}{h} (k - 2k^3 + k^4) \dots \dots \dots (32)$$

in which  $kL$  = distance from support. Incidentally, Eq. 32 is the same as Eq. 16a, since  $j = 2k$ . This formula can be employed with only a slight error for the preliminary proportioning of a circular arch of usual proportions.

That no greater precision is necessary is indicated by the use of Table 6 as illustrated in Table 7. Values for the maximum influence ordinate, as obtained

TABLE 7.—COMPARISON OF INFLUENCE ORDINATES IN TABLE 6 WITH COMPUTATIONS BY EQ. 32

No.	Arch (see Table 6)	MAXIMUM INFLUENCE ORDINATE			PERCENTAGE ERROR	
		"Reduced"	Actual	Eq. 32	"Reduced"	Eq. 32
1	Tyngsborough	1.113	1.104	1.114	+1.0	+1.0
2	Cottage Farm	0.852	0.838	0.813	+1.7	-3.0
3	Hell Gate	0.849	0.841	0.813	+1.0	-3.4

by the use of Eq. 32, are well within the required accuracy for preliminary design, compare favorably with the values obtained by the use of the author's reduced equations, and are more simple to compute. Preliminary determinations within 3% are fairly close. Variation of details may account for greater differences.

At its best, the method appears to be directly applicable only to solid concrete ribs of such shapes as the author (see paragraph following Eq. 16b) has selected "primarily because of their mathematical simplicity." It is clear that for steel ribs composed of shapes and plates, where changes in sections are abrupt, the mathematics would become so involved as to make the method too cumbersome for practical use. The author claims that the method is also applicable to arch trusses and to three-span continuous trusses; it is yet to be demonstrated, however, just how such application would be useful in any but very simple cases, where any relation to a practical design problem would be purely coincidental.

It may be pertinent to state that the actual time required for the analysis of any usual structure is a very small part of the total time required to prepare

<sup>7</sup>"The Theory and Practice of Modern Framed Structures," by J. B. Johnson, C. W. Bryan, and F. E. Turneaure, 9th Ed., John Wiley & Sons, Inc., New York, N. Y., 1910-1916.

designs, plans, and specifications. Hence, the difference between one or two, or even three, trial designs reflects the total effort only in a minor way. Each designer as a rule develops his own peculiar approach to a problem and recognizes "short cuts" as his experience broadens. The quickest design method for any structure is the method that the designer knows and understands best.

Of late entirely too much emphasis seems to have been placed on methods and not enough on the physical meaning of the mechanics of calculations. Thus, as an example, the significance of Eq. 8a, for a horizontal reaction of a two-hinged arch, is simply that the horizontal reaction is equal to the horizontal movement of an arch end due to superimposed loads (if the end were free to move) divided by the similar movement caused by a unit load applied at that end. Such a definition is easily understood and the problem then is one of determining these movements of the arch end. Unfortunately, the true meaning of Eq. 8a can easily be befogged by bending moments, ordinates, and differential elements—the actual significance of the meaning being lost in the shuffle.

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## DISCUSSIONS

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### CONVERSION OF KINETIC TO POTENTIAL ENERGY IN FLOW EXPANSIONS

#### Discussion

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BY F. T. MAVIS

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F. T. MAVIS,<sup>13</sup> M. Am. Soc. C. E.<sup>13a</sup>—In a direct fashion by patient compilation and analysis of statistical data, Professor Kalinske has presented a paper which is based on a staggering number of photographs.

Whenever water decelerates in a conduit under conditions of steady flow, there is a conversion of kinetic to potential energy. This conversion is not made without loss in excess of that induced by the surface of the conduit. The losses which accompany this conversion of energy have been studied intensively and the methods of study can be classified broadly into two categories:

- (1) Quantitative experiments on a particular installation or model from which over-all losses incident to the conversion can be deduced; and
- (2) Mathematical analyses of general and often abstract motions of an idealized fluid.

Category (1) is the usual engineering method. It can lead to valid engineering deductions and can contribute much directly to the fundamental understanding of the mechanism of energy conversion. Both approaches to the study of the conversion of kinetic to potential energy have yielded significant contributions.

The engineering approach is usually to seek a particular solution of a problem—leaving to others the related problems of correlating, generalizing, and rationalizing the data into a general pattern. The mathematical approach usually begins with an idealized problem, leads to a general hypothesis, and leaves to others the task of applying these generalizations to particular cases. Too often these two approaches to the problem—springing from opposite sides of a chasm of unknowns—do not meet to form a well-integrated structure of theory and experiment. The investigation reported by Professor Kalinske was directed toward studying the internal mechanism of energy conversion in con-

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NOTE.—This paper by A. A. Kalinske was published in December, 1944, *Proceedings*.

<sup>13</sup> Prof. and Head, Civ. Eng. Dept., Carnegie Inst. Tech., Pittsburgh, Pa.

<sup>13a</sup> Received by the Secretary January 6, 1945.

duits with straight axial expansions and toward forging another link in this integrated rational structure.

The purpose of this discussion is (a) to classify the elementary phenomena of conversion of kinetic to potential energy in hydraulic engineering, and (b) to present selected bibliographic data which may suggest the course of further laboratory studies and illumine the weakness or strength of the mathematical theory.

Table 5 classifies the fundamental problems of the hydraulic conversion of kinetic to potential energy and the significant geometrical variables for each case. Common to all the problems listed are such physical variables as rate of discharge, roughness of conduit, state of motion, head loss, and velocity distribution. There are but two major classes of flow. Within the reach of the conduit under consideration the flow is either fully confined by the walls of the conduit (Table 5(b)), or not fully confined (Tables 5(c) and 5(d)), in which case there is a free water surface in the reach. The interplay of unbalanced forces and accelerations of liquid elements is maintained in the first case by changing pressures along fixed boundaries, and in the second case by an aperiodic pulsating free water surface or by a complex zone of air-entrained water.

A conduit which fully confines the flow within a reach under consideration may induce deceleration of the liquid by reducing the velocity of flow or by changing its direction. There are four primary elements:

1. A straight axial expansion which is represented in its simplest form by the lower leg of a venturi meter, by a straight, conical draft tube, and by similar diffusers;<sup>14</sup>
2. An axial-to-radial expansion of which the spreading draft tube is an example;<sup>15,16,17,18,19,20,21</sup>
3. An elbow in which the flow can be decelerated by changing the direction of flow only, or by a combination of changes in direction and changes in average velocity. Elbows may, therefore, be subdivided further into—

- (a) Prismatic elbows in which the cross section and shape of the conduit remain unchanged;

<sup>14</sup> "Report of the Committee on Hydrodynamics," by Hugh L. Dryden, Francis D. Murnaghan, and H. Bateman, *Bulletin No. 84*, National Research Council, Washington, D. C., 1932, pp. 469-481 (bibliography, 68 entries).

<sup>15</sup> "Report of the Committee on Hydrodynamics," by Hugh L. Dryden, Francis D. Murnaghan, and H. Bateman, *Bulletin No. 84*, National Research Council, Washington, D. C., 1932, pp. 481-492 (bibliography, 58 entries).

<sup>16</sup> "A Study of the Hydrodynamics of Spreading Draft Tubes," by Andreas Luksch, thesis presented to the State Univ. of Iowa, Iowa City, in June, 1935, in partial fulfillment of the requirements for the degree of Doctor of Philosophy (bibliography, 52 entries).

<sup>17</sup> "Hydraulic Tests of Small Diffusers," by F. T. Mavis, Andreas Luksch, and Hsi-Hou Chang, *Bulletin No. 13*, Studies in Engineering, Univ. of Iowa, Iowa City, 1938.

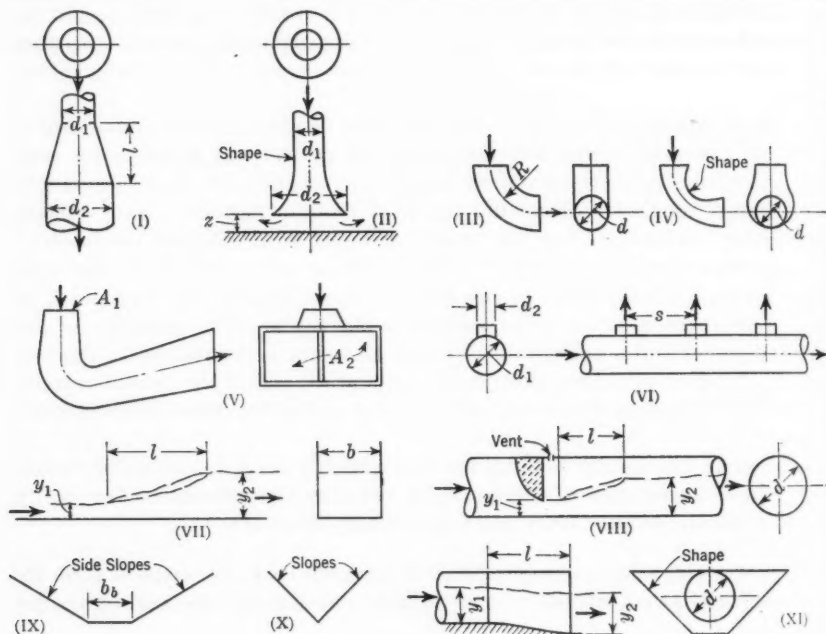
<sup>18</sup> "Flow Characteristics in Elbow Draft-Tubes," by C. A. Mockmore, *Transactions, Am. Soc. C. E.*, Vol. 103 (1938), pp. 402-464.

<sup>19</sup> "Bibliography of Important Papers on Draft Tubes Since 1913," National Bureau of Standards, U. S. Dept. of Commerce, Appendix I to report on investigation of draft tubes conducted for the Tennessee Valley Authority, June 30, 1935 (bibliography, 78 entries).

<sup>20</sup> "Flow of Water Around Bends in Pipes," by David L. Yarnell and Floyd A. Nagler, *Transactions, Am. Soc. C. E.*, Vol. 100 (1935), pp. 1018-1043.

<sup>21</sup> "Flow of Water Around 180-Degree Bends," by David L. Yarnell and Sherman M. Woodward, *Technical Bulletin No. 526*, U. S. D. A., 1936 (bibliography, 19 entries).

TABLE 5.—ELEMENTS OF HYDRAULIC CONVERSION OF KINETIC TO POTENTIAL ENERGY



(a) SKETCHES OF PIPE AND CHANNEL SECTIONS

ELEMENT		Geometrical variables (see sketches in Table (a))
Description	No.	

(b) FLOW FULLY CONFINED		
Straight axial expansion . . . . .	i	Diameters $d_1$ and $d_2$ , length $l$ , and the shape
Axial to radial expansion . . . . .	ii	Diameters $d_1$ and $d_2$ , the shape, clearance $z$ , and the inserts or the vaning
Prismatic elbow . . . . .	iii	Diameter $d$ , radius $r$ , and the central angle
Bisymmetrical elbow . . . . .	iv	The shape and the vaning
Expanding elbow . . . . .	v	Areas $A_1$ and $A_2$
Manifold . . . . .	vi	Diameters $d_1$ and $d_2$ , spacing $s$ , and the shape of the passages

(c) FLOW NOT FULLY CONFINED; SUPERCRITICAL INITIAL VELOCITY, $V_0$		
Hydraulic Jump—		
Rectangular conduits . . . . .	vii	Depths $y_1$ and $y_2$ , breadth $b$ , and length $l$
Circular conduits . . . . .	viii	Diameter $d$ and the venting
Trapezoidal conduits . . . . .	ix	Bottom width $b_0$ and the side slopes
Triangular conduits . . . . .	x	Side slopes (special case of sketch ix)
Jets . . . . .		(Surface, plunging, and partly confined)

(d) FLOW NOT FULLY CONFINED; SUBCRITICAL INITIAL VELOCITY, $V_0$		
Transitions . . . . .	xi	Depths $y_1$ and $y_2$ , diameter $d$ , length $l$ , and the shape

- (b) Bisymmetrical elbows in which the cross section or shape is symmetrical with reference to two planes—one in the plane of the elbow axis and the other normal to that plane and bisecting the straight approach and outlet conduits; and
- (c) Expanding elbows represented by elbow draft tubes.

4. A manifold which may be represented by the filling and emptying conduit system of the more common American type of navigation locks. (In the sense that the water is discharged into a lock chamber through submerged ports during the filling stage, the manifold may also be classified as an example of a partly confined jet.<sup>23</sup>)

Decelerating flow in a conduit with a free surface (flow not fully confined by the walls of the conduit) has two major subdivisions:<sup>23, 24, 25, 26, 27</sup>

I. If the velocity in a reach is supercritical (velocity head greater than half the hydraulic mean depth), the water either passes through an hydraulic jump, or continues in some form of jet. The hydraulic jump has been studied intensively in rectangular conduits, circular conduits, trapezoidal conduits, and conduits of other sections. Surface waves and plunging jets have been studied in open channel flow and submerged weirs chiefly.

II. If the approach velocity in an expanding conduit is subcritical (velocity head less than half the hydraulic mean depth), the hydraulic problem of the transition becomes primarily a matter of backwater curves. This topic has received extensive experimental and mathematical study. Practical problems in this category can be handled quite satisfactorily by combinations of laboratory study and conduit analysis.

Essentials in the study of energy conversion in flowing water are Newton's laws of motion, a complete physical concept or hypothesis, exact definitions of terms, rational induction and deduction, and objective comparisons and experiments. The engineer has usually approached the study with the objective of determining over-all losses of energy rather than with the purpose of studying the internal mechanism of energy conversion. Many guide posts to engineering judgment have been established on the basis of so-called one-dimensional solutions of problems in hydraulics. Five illustrations of "one-dimensional" approaches to problems involving conversion of kinetic to potential energy in flowing water are sketched in the following paragraphs:

A. Newton's second law of motion expresses the relationship between the unbalanced force acting on a particle and its corresponding acceleration.

<sup>23</sup> "Lock Manifold Experiments," by Edward Soucek and E. W. Zelnick, *Proceedings*, Am. Soc. C. E., October, 1944, pp. 1255-1274.

<sup>24</sup> "Hydraulics of Open Channels," by B. A. Bakhmeteff, McGraw-Hill Book Co., Inc., New York, N. Y., 1932.

<sup>25</sup> "The Hydraulic Design of Flume and Siphon Transitions," by Julian Hinds, *Transactions*, Am. Soc. C. E., Vol. 92 (1928), pp. 1422-1485.

<sup>26</sup> "Hydraulic Jump in Trapezoidal Channels," by C. J. Posey and P. S. Hsing, *Engineering News-Record*, December 22, 1938, pp. 797-798.

<sup>27</sup> "Determining the Energy Lost in the Hydraulic Jump," by J. C. Stevens, *ibid.*, June 4, 1925, pp. 928-929.

<sup>28</sup> "Hydraulics of Steady Flow in Open Channels," by Sherman M. Woodward and Chesley J. Posey, John Wiley & Sons, Inc., New York, N. Y., 1941.

Precisely consistent relationships can be expressed between the work done on a particle and the change in its kinetic energy, or between the impulse of a force acting on a particle and the change in its linear momentum. Consistently applied, the principles of force and acceleration, work and energy, and impulse and momentum lead to identical conclusions. Evidence that this fact is sometimes misunderstood is the inference that Bernoulli's theorem (essentially work and energy) "applies" in some cases and that the momentum principle "applies" in others. When all factors are considered in any given problem, identical conclusions follow the logical application of the principle of work and energy or the principle of impulse and momentum. Students of hydraulics sometimes do not realize fully that apparent "differences" are but logical deductions which follow from differences in viewing the complete physical picture or concept.

*B.* The principle of work and energy applied to the flow of a liquid in a stream tube leads directly to Bernoulli's theorem. If real water flows in a real conduit, the walls of the conduit exert a drag on the flowing liquid which opposes the motion. In mathematical derivations and algebraic manipulations, if the physical pictures are incomplete or inconsistently simple, "other factors" are sometimes introduced presumably as logical conclusions rather than as hypotheses whose validity must eventually be tested by experiments.

*C.* The principle of impulse and momentum applied to decelerating flow of water in a level open conduit leads to an expression for depths of flow adjacent to the hydraulic jump. Simple assumptions lead to an expression for head loss in terms of alternate or sequent depths of flow which is in substantial agreement with laboratory and field observations. Suffice it to state that, if the principle of impulse and momentum is logically applied to accelerating flows of water in a level open channel, the valid conclusion must be reached that the water surface promptly becomes level as a consequence of the basic definition of a fluid. On any horizontal plane the fluid cannot indefinitely resist shearing stresses which must be set up by unbalanced pressures parallel to the plane.

*D.* The principle of impulse and momentum applied to decelerating flow in a  $90^\circ$  expansion leads to differences in pressure above and below the expansion which depend on the assumed pressure distribution on the face of the expanding section. Reasonable assumptions of this pressure distribution lead to computed head losses which are in substantial agreement with laboratory observations.

*E.* The principle of impulse and momentum applied to "nonresilient impact" of two particles moving in the same direction with different initial velocities leads to an expression for head loss which is identical with the expression for head loss in a  $90^\circ$  expansion in a closed conduit. In the over-all analysis of head loss in a sudden expansion, nothing need be assumed as to the detailed mechanism of conversion. Particles at the boundary of the jet approaching the expansion come in contact with slower moving particles to emerge—after mixing—with reduced velocity. The term "inelastic impact," often used to designate this phenomenon, implies an inexact concept since a

confined liquid is "elastic" in the sense that it has a well-defined bulk modulus of elasticity. Individual globules of an unconfined liquid are not resilient, however, in the sense that mechanical energy can be stored in an unconfined globule by virtue of its elastic deformation. Hence, the term nonresilient impact seems more descriptive. This nonresilient impact of liquid masses may be viewed as a physical basis for analyzing the conversion of kinetic to potential energy, and it may throw added light on the author's second conclusion that "The maximum total turbulence energy is a small part of the total energy change taking place."

Professor Kalinske has outlined briefly certain concepts relating to turbulence and he has referred to selected literature. The engineer will doubtless feel a void between these concepts and their application to even an elementary engineering situation. If statistical methods of studying the phenomena of turbulence and the nonrecoverable components of kinetic energy in a stream are to be applied to a given conduit, one must know the primary pattern of flow through the conduit, the intensity of turbulence at any given point in the conduit, and the scale of turbulence at the corresponding point. At present, if not inherently, these data must be obtained experimentally and at the expense of considerable refinement in laboratory techniques. If the data are obtained from model tests with the intention of translating results from one conduit to another, it follows that the two conduits must be geometrically similar, the intensity of turbulence at corresponding points in the two conduits must be identical, and the scale of the turbulence in the two conduits must be geometrically similar.

The practical significance of the statistical approach to fluid turbulence in the study of the conversion of kinetic to potential energy has not yet reached far beyond the laboratory and the computer's desk. The engineer certainly cannot look to the analyst of turbulence problems for the design of a diffuser, a draft tube, or the hydraulic system for a lock. However, if the basic theory of fluid turbulence prompts the experimenter and the engineer to look more critically into the effect of baffles and the internal similarity of flow patterns, the mathematical approaches—as yet of limited direct application—may eventually make their effects felt in more important problems of hydraulic engineering analysis and design.

The study described in this paper was initiated by the writer in 1936 at the invitation of J. C. Stevens, President, Am. Soc. C. E., chairman of the Society's Committee on Hydraulic Research, and progress reports have been abstracted periodically in *Civil Engineering*. Between 1936 and 1939 a bibliographic study was completed by Andreas Luksch, graduate assistant in hydraulics at the State University of Iowa, and selected tests were conducted by Mr. Luksch; by Edward R. Van Driest, Assoc. M. Am. Soc. C. E., and Arthur R. Luecker, Jun. Am. Soc. C. E., research assistants in mechanics and hydraulics; and by Hsi-Hou Chang, graduate student. In 1939 and subsequently the work on the project was directed by Professor Kalinske.

Credit for prior development of the photographic technique which played such a significant rôle in the studies reported by Professor Kalinske is due to Edgar E. Ambrosius, John C. Reed, and Henry F. Irving<sup>23</sup> who pursued a related program of basic studies in the hydraulics laboratory of the University of Illinois at Urbana which was under the direction of M. L. Enger, M. Am. Soc. C. E.

<sup>23</sup> "Study of the Flow of Water Through a Glass Pipe," by Edgar E. Ambrosius, John C. Reed, and Henry F. Irving, preprinted papers and program, Aeronautic and Hydr. Divs., A.S.M.E., Summer Meeting, June 19, 20, and 21, 1934, George Reproduction Co., San Francisco, Calif., pp. 83-89.

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# AMERICAN SOCIETY OF CIVIL ENGINEERS

Founded November 5, 1852

## DISCUSSIONS

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### STABILITY AND STIFFNESS OF CELLULAR COFFERDAMS

#### Discussion

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BY DEAN P. TSAGARIS

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DEAN P. TSAGARIS,<sup>6</sup> M. AM. SOC. C. E.<sup>6a</sup>—Comparatively little has been written or published on the rational design procedure and the behavior exhibited by this type of cellular cofferdam construction. Therefore, assumptions, analyses, and final judgment must be based on actual experience and observation. The paper constitutes a valuable and significant contribution to the engineering profession on this subject. As the author states, tests should be made, but results should be used with discretion, since they have often proved inconclusive.

The following comments and discussion are confined to cofferdams in which piling is driven to rock through an overburden varying from several feet to 50 ft in depth. Experience is largely drawn from the many cofferdams designed and built by the Tennessee Valley Authority (TVA) for the purpose of enclosing construction operations.

The inherent economic advantages in using circular cells with connecting arcs are due primarily to simplicity of construction and re-use on subsequent stages and projects. Furthermore, cells can be filled independently without interruption, and possible interlock failure will be localized in one cell. As a general rule, if a cofferdam does not exceed 50 ft in height and is carefully designed for sliding and interlock, allowing for saturation of fill within the cells, the resulting construction will be satisfactory and safe.

Designing on an "equivalent width" basis (and a length of unity) was found to be sufficiently accurate after a comparison was made with the results obtained from using a typical cell as a unit. This expedient obviously simplified calculations greatly.

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NOTE.—This paper by Karl Terzaghi was published in September, 1944, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1944, by G. G. Greulich, and Raymond P. Pennoyer.

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<sup>6a</sup> Received by the Secretary January 17, 1945.

Because of the great height of the Kentucky cofferdam (100 ft above rock) and of the magnitude of this undertaking as a whole, very careful consideration was given to the following factors: (a) Saturation of fill material within the cells; (b) interlock tension; and (c) internal resistance to shear. In view of this concern, the writer cannot agree with the statements made on the subject of internal shear, which read (see second paragraph following Eq. 8): "\* \* \* before this failure [by overturning] occurs, the cofferdam is likely to fail from another cause which has never received any attention" and (see second paragraph preceding Eq. 28a) "\* \* \* a potential source of failure by shear along vertical planes (Eq. 19b) which has been ignored in the past \* \* \*."

(a) *Saturation of Fill Material Within the Cells.*—On all cofferdams built by TVA, a mixture of sand and gravel with little silt content was used to obtain as much free drainage as possible. Filling was done by the hydraulic process, except for a few cells built in the dry, such as tie-ins. When the first cofferdams were constructed, the need for dependable information on saturation was realized. For purposes of study, several rows of pipes were provided within the cells, and observations were taken, after unwatering, on the level of water resulting from both a rising and receding river.

In general, results disclosed a rather flat gradient across the cells, beginning from a point somewhat below the outside river elevation. No complete drainage was evidenced in spite of the customary weep holes on the dry side. In any case, much depends on the care exercised in the field and the extent of the provisions made to ameliorate saturation.

A saturation line with an approximate slope of 1 on 2 starting from the outside water elevation was considered satisfactory, and slopes as steep as 1 on 1 were adopted where special care was taken to keep cells well drained. Hence, all calculations were made accordingly on subsequent jobs, using 110 lb per cu ft for material above the saturation line, and 65 lb per cu ft for material below the saturation line. As a result, all cofferdams required a berm on the dry side to satisfy the aforementioned requirements. This does not necessarily mean that cells cannot be made wide enough to avoid berms, provided they check in every other respect and are economical. A horizontal line, at an elevation so chosen as to represent the average expected condition of saturation, should serve just as well and at the same time simplify computations.

(b) *Interlock Tension.*—This is one of the most important features in cellular cofferdam design and should be kept within safe limits. The practice of computing pressures on a straight-line variation all the way to rock is fallacious since cells almost invariably exhibit a characteristic bulge about one quarter to one third of the way up from the base. Driving apparently forces the steel sheet piling into rock sufficiently to prevent any appreciable movement at this point. When berms are used a similar condition exists, except that the overburden or the berms serve to restrain the piling. In the light of this phenomenon, the writer believes that pressures from which interlock tension is computed should be considered maximum at three quarters of the way down from the top in cases without berms. If berms are used, however, maximum pressures will occur at a somewhat higher position.

With the improvement of piling and the ability of the manufacturers to guarantee higher strengths, interlock failures have been very few and are due primarily to construction operations such as difficult driving. In such instances "brooming" and opening of interlocks have resulted near the bottom, but repairs can be made easily, especially in cases where no berms are used. All possible conditions should be investigated to determine maximum interlock tension, particularly during filling operations when the river elevation is low, because this condition most often controls.

The author does not mention the amount of tension contributed by the connecting arcs to the existing main cell tension. This has been found considerable, and should be neither neglected nor computed as a free force acting on the main cell through the tees or wyres. The neglect cannot be excused, and computation as a free force does not consider the passive resistance of the fill within the connecting arcs, which resists a large percentage of the tension before it can find its way into the main cell.

(c) *Internal Resistance to Shear.*—The possibility of shear failure has been recognized by TVA engineers concerned with cofferdam design, and investigations in this connection have been thorough. No rigorous mathematical analysis is justified, however, when friction coefficients for fill material, the actual pressure against the inside of the piling, and the resistance offered by interlocks are so variable. Very ably, Professor Terzaghi explains the implications involved in computing actual shear and evaluating the resistance to this shear along a selected critical plane.

Relatively small horizontal deflections are capable of mobilizing high pressures within the fill, especially against the inside surface of the piling, with equally large resistances. This tendency will continue as the lateral load increases provided the cofferdam is still safe against interlock failure, sliding, and overturning. Furthermore, if slippage occurs between sand and gravel particles, the new configuration of material may conceivably offer a greater resistance to shear than the results of computations indicate. Readjustment in each case is associated with planes of weakness, but other contact surfaces not so highly stressed will come into play to relieve the situation until horizontal movement due to slippage stops. This behavior transcends the realm of hypothesis; otherwise the paradox of nonfailure for cases with factors of safety less than unity cannot be explained. Since all cofferdams built by TVA were temporary, a factor of safety of 1.25 for shear, sliding, and interlock tension was considered quite adequate.

Briefly, the method used to compute total shear in any vertical plane, and the resistance, followed the general procedure described by the author, except that it was recognized that active pressure within the material was augmented by the effect of external horizontal forces tending to compact the fill confined in the cells. A friction coefficient of 0.55 was used. To the foregoing, the resistance against interlock slippage was added, using a coefficient of friction of 0.3, which gave the total effective resistance to shear. Under the most unfavorable conditions, the smallest factor of safety was 1.2. It is felt, however, that the coefficients of friction for both internal shear and slipping of interlock were quite conservative.

*Recommendations.*—To summarize, the writer would like to offer six recommendations, as follows:

1. Circular or clover-leaf type cells with connecting arcs should be used;
2. Selected free draining fill material, such as sand and gravel, will be found best;
3. Computations should be based on saturated fill below a line with an approximate slope of two horizontal to one vertical (for simplicity a horizontal line may be adopted at an elevation which represents average conditions);
4. Shear resistance should be considered and computed on a logical and rational basis;
5. Interlock tension must be kept within safe limits and the modifying effects of the characteristic bulge in sheet piling should be recognized; and
6. Grouting of the foundation rock will result in a tighter and safer cofferdam.

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# AMERICAN SOCIETY OF CIVIL ENGINEERS

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## DISCUSSIONS

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### COLUMN FORMULAS

#### Discussion

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BY L. E. GRINTER, GEORGE WINTER, AND JONATHAN JONES

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L. E. GRINTER,<sup>6</sup> M. AM. SOC. C. E.<sup>6a</sup>—Since the column is probably the most critical structural member, it is well for the structural engineer to review, at intervals, the background of procedures for column design. Design of columns has long been associated with the use of a "column formula" and, therefore, the author's excellent paper again stimulates the profession to re-examine its methods.

Test results for columns of varying slenderness, when plotted on a graph of failure stress versus slenderness ratio, do not fall along a single line but represent a broad band even when the tests were all performed in one laboratory. This is true because of the virtual impossibility of producing a centered load or a perfectly homogeneous column, particularly if it must be fabricated from parts. No doubt the spread of failure loads would be much greater for columns in actual structures where soil movements, temperature movements, and strains of fabrication have an influence upon resistance to failure.

The choice of a column formula, therefore, is something of a psychological or philosophical matter rather than a strictly scientific decision. A straight line, a broken line, a parabola, or a reversed curve can each be chosen to represent test results, probably with equal satisfaction and accuracy, within the common working range of slenderness ratios. If this is true, then the choice of a formula should be influenced by another consideration: What column formula will produce the best results in terms of design when placed in the hands of the ordinary designer who is not always a highly competent structural engineer?

There are two mistakes that can be made in the use of a column formula. The formula itself may be misinterpreted; and this mistake will become more probable as the formula becomes more complex. For this reason either the Kreüger formula or the Aarflot formula mentioned by the author could become a source of errors in design. The second mistake is in extending the use of the

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NOTE.—This paper by William R. Osgood was published in December, 1944, *Proceedings*.

<sup>6</sup> Vice-Pres. and Dean, Graduate School, Illinois Inst. of Technology, Chicago, Ill.

<sup>6a</sup> Received by the Secretary January 4, 1945.

formula beyond its specified range. This can be counteracted by emphasizing the necessary limitations on the formula or by selecting a formula of the kind recommended by the author for which extrapolation beyond test results will give reasonably safe results.

Perhaps it is unfortunate, but, nevertheless, much design work is actually performed by persons of limited competency in structural design. For them the straight-line formula or the Rankine formula is the reasonable limit of useful complexity. Since no greater accuracy can be obtained by use of a more complex formula, the advantage of adding terms is not very clear. The best form of printed specification and one which would meet all objections would be a complete statement of the design formula to cover each range of slenderness ratio. The only mistake that has been common in specification writing has been to cover only the normal range and simply to neglect entirely the upper and lower ranges with resulting confusion in the minds of many persons who must, and do, perform design work.

GEORGE WINTER,<sup>7</sup> M. AM. SOC. C. E.<sup>7a</sup>—Until a decade or two ago practical column formulas were exclusively of the empirical type. Mathematical expressions were simply chosen to approximate, as closely as possible, the results of a greater or smaller number of column tests. Little attention was paid to the physical significance of these formulas, to correct representation of end fixities, manufacturing peculiarities, and sometimes even to physical characteristics of the material. Since then the development has been along more rational lines. Methods were developed to account for column strength by considering, in a rigorous analytical way, properties of material, end fixity, initial shape of the member, manner of load application, etc. Two types of such methods have evolved:

(a) The one designated by the author as double-modulus theory which takes account of the deviation from the straight line of the stress-strain curves, particularly at higher stresses. This method proved rather cumbersome in its application and, as the author states, is of only limited practical consequence.

(b) The other takes account of the fact that the strength of a column depends to a great extent on its initial shape and the manner of load application. Initial deviations from straightness, eccentricities of loading, and other factors which might be termed imperfections were found to decrease the strength of columns considerably, particularly in the range of low and medium slenderness ratios. The secant formula is of this latter type as is the method proposed by D. H. Young<sup>8</sup> in 1934.

The great advantage of methods (b) consists in the fact that any term entering such a column formula has a definite physical significance and the entire formula is based on rigorous elastic theory rather than merely representing a necessarily limited number of test results. In the hands of a discriminat-

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<sup>7a</sup> Received by the Secretary January 2, 1945.

<sup>8</sup> "Rational Design of Steel Columns," by D. H. Young, *Transactions, Am. Soc. C. E.*, Vol. 101 (1936), p. 422.

ing designer such a formula can be adapted to almost any set of particular circumstances, such as eccentricities of loading, imperfections of shaping, etc., which is entirely impossible with the purely empirical type of formula. Too much reliance on the latter actually may lead to unsafe design, inasmuch as a column, with larger amounts of eccentricity or crookedness than were inherent in the test specimens, actually fails at loads often considerably below those given by the empirical formula. After many years of extensive experimental research the Society's Special Committee on Steel Column Research<sup>9</sup> recommended a formula of this rational type—the secant formula—as the basis for design specifications. It found this formula not only to be rational and definite in the physical significance of its mathematical terms, but also established its excellent agreement with the results of an unusually large number of tests. The formula is directly adaptable to any given properties of material, eccentricity of loading, imperfection of initial shape, factor of safety, and other special conditions. The design specifications for railway bridges of the American Railway Engineering Association (A.R.E.A.) are presently based on this formula and, taking advantage of its versatility, are adapted not only to standard structural steel but also to a number of special steels used in bridges.

The author's proposed formulas are exclusively of the first, purely empirical type and, therefore, in the writer's opinion, represent a definite step backward in the direction of formal expressions merely adapted to a given set of tests but devoid of fundamental physical meaning. Therefore they are not applicable to any structures except those which duplicate the test setup in all significant details. They are purely mathematical expressions representing S-shaped curves with zero slope at the lower end and zero asymptote for large values of  $l/i$ ; that is, they merely have the formal mathematical characteristics of all column curves without being capable of physical interpretation.

In the writer's opinion the desirable development in this field lies in the direction of rational column formulas of the type mentioned. For standard design conditions the way a column curve was originally determined may be of little consequence. Under nonstandard circumstances—such as columns of unusual alloys, special loading conditions (eccentricities), and particular fabricating methods—which may involve either larger or smaller imperfections of shape than are commonly met with, formulas of the rational type give considerable guidance to the designer even without, or with very limited, test evidence. The author's formulas, to be applicable, would require elaborate tests in each of these special cases. It is precisely for this reason (to take care of unusual situations) that the A.R.E.A., in an appendix to its specifications, included the complete secant formula with the physical significance of all its terms explicitly stated. From his own design experience with structures of nonstandard material and nonstandard but at least approximately known imperfections, the writer can testify to the reliability of this method which, in some cases he dealt with, was very successfully verified by a small number of full-scale confirmatory tests.

The secant formula, and others of the same type, permit the engineer to predict the maximum fiber stress under a given load for a column of any

<sup>9</sup> Transactions, Am. Soc. C. E., Vol. 98 (1933), p. 1376.

material provided the following characteristics are known with reasonable accuracy: Properties of material, end conditions, eccentricities of loading, and initial deviations from straightness (crookedness). By equating this maximum fiber stress to some stress considered as dangerous (such as the yield point for steel) and by applying a suitable factor of safety, a practical column formula is immediately derived which covers the entire range of slenderness ratios. The reason that, despite these advantages, the explicit secant formula has not become too popular with designers lies in the fact that it is rather unwieldy mathematically. It requires the use of trigonometric tables and cannot be solved explicitly for the working stress for a given slenderness ratio; that is, it necessitates in each case the drawing of a complete column curve. This practical difficulty is easily overcome by developing general column formulas of the standard familiar type (Rankine and Johnson formulas), which are made to fit very closely the values of the exact secant formula.

The general form of the secant formula is

$$\left(\frac{P}{A}\right)_w = \frac{\frac{\sigma_y}{\eta}}{1 + a \sec \left[ \frac{k l}{2 i} \sqrt{\frac{\eta}{E}} \left(\frac{P}{A}\right)_w \right]} \dots \dots \dots (22)$$

in which (supplementing the notation of the paper):  $\left(\frac{P}{A}\right)_w$  = working stress in compression;  $\sigma_y$  = limiting (that is, dangerous) value of extreme fiber stress, generally the yield point of the material;  $\eta$  = factor of safety; and  $a$  = "degree of imperfection"; that is—

$$a = \frac{(e + d) c}{i^2} \dots \dots \dots (23)$$

in which  $e$  = eccentricity of loading;  $d$  = maximum deviation from straightness (crookedness) of column; and  $c$  = distance from center of gravity to outer fiber of cross section.

Accepted values for these constants are:  $\eta = 1.7$  to  $1.8$  for bridges and  $1.5$  to  $1.6$  for buildings;  $a = 0.25$  for standard manufacturing conditions and centered connections; and  $k = 0.75$  for riveted and welded connections and  $0.875$  for pin connections.

A very close approximation of the values of the secant formula in the low and medium range of  $l/i$  is obtained by the expression

$$\left(\frac{P}{A}\right)_w = \frac{\sigma_y}{n_s} - \frac{\sigma_y^2 k^2}{4 n_s \pi^2 E} \left(\frac{l}{i}\right)^2 \dots \dots \dots (24)$$

in which

$$n_s = \eta (1 + a) \dots \dots \dots (25)$$

Eq. 24 is of the Johnson parabolic type familiar to designing engineers. Most column specifications, such as those of the American Institute of Steel Construction (A.I.S.C.) and the American Railway Engineering Association (A.R.E.A.), are written in terms of the Johnson formula for the low and

medium range of  $l/i$ . The values obtained from Eq. 24 are extremely close to those of the secant formula, Eq. 22, for all values of the working stress down to

$$\left(\frac{P}{A}\right)_w = 0.5 \left(\frac{P}{A}\right)'_w = 0.5 \frac{\sigma_y}{n_s} \dots (26)$$

in which  $\frac{\sigma_y}{n_s}$  is the working stress at  $l/i = 0$ . The limiting value of  $l/i$  at

which  $\left(\frac{P}{A}\right)_w = 0.5 \left(\frac{P}{A}\right)'_w$  is

$$\left(\frac{l}{i}\right)_{\text{limit}} = \frac{\pi}{k} \sqrt{\frac{2E}{\sigma_y}} \dots (27)$$

For values of  $l/i$  larger than those given by Eq. 27, the Johnson curve deviates increasingly from the secant curve and, therefore, should no longer be used for design purposes. It will be found that for structurally useful alloys the value

of  $\left(\frac{l}{i}\right)_{\text{limit}}$  ranges from about 80 to 160, depending on mechanical properties.

(The use of  $0.5 \left(\frac{P}{A}\right)'_w$  as the end point for the Johnson formulas was first suggested to the writer by Prof. J. H. Cissel, M. Am. Soc. C. E., of the University of Michigan, Ann Arbor, in a discussion concerning general questions of column design.)

For slenderness ratios larger than  $\left(\frac{l}{i}\right)_{\text{limit}}$  a very close approximation of the secant formula is obtained by the expression

$$\left(\frac{P}{A}\right)_w = \frac{B}{1 + \frac{1}{C} \left(\frac{l}{i}\right)^2} \dots (28)$$

in which

$$C = \frac{500,000 \pi^2 E \eta a}{250,000 k^2 \eta \sigma_y - 2 n_s \pi^2 E} \dots (29a)$$

and

$$B = \frac{\sigma_y}{2 n_s} + \frac{1 \pi^2 E}{C k^2 n_s} \dots (29b)$$

Eq. 28 is adjusted to give working stresses extremely close to those of the secant curve up to values of  $l/i = 400$  and even higher. This was achieved

by making the value of  $\left(\frac{P}{A}\right)_w$  from Eq. 28 equal to (Euler value)/ $\eta$  at  $l/i = 500$ . At such large slendernesses the values of the secant formula are almost identical with those of the Euler equation which results in the close approximation of Eq. 28 with the secant values.

It is seen that Eqs. 24 and 28 are given in the same general terms as is the secant formula ( $\sigma_y$ ,  $E$ ,  $\eta$ ,  $k$ ,  $a$ ) and result in column formulas of the type

familiar to the designer, that is, the Johnson formula for small and medium slenderness ratios and the Rankine formula for large slenderness ratios. Thus, for a given material and given conditions, design formulas, Eqs. 24 and 28, are computed once and for all by substituting appropriate values for the constants. The resulting formulas are then used in the same simple manner as any standard column formula in current codes. In fact, they are of the same form as those used in the A.I.S.C. specifications. In this manner the use of the arithmetically cumbersome secant formula is entirely eliminated, whereas all the advantages are retained. The formulas are adaptable to any type of material of known mechanical properties, which is likely to become advantageous with the increasing use of various ferrous and nonferrous alloys for structural purposes. The approximation is extremely close for all values of  $a$  from 0.1 to about 0.6; that is, in the practically important range.

The use of these formulas and the excellent approximation they furnish to the values of the secant formula are illustrated in the following two examples:

Example 1.—Consider a column of structural grade steel, under the following design conditions:  $\sigma_y = 33,000$  lb per sq in.;  $E = 29,500,000$  lb per sq in.;  $\eta = 1.76$ ;  $a = 0.25$ ; and  $k = 0.875$ . These data are precisely the same as those assumed for pin-ended members in the A.R.E.A. Specifications for Steel Railway Bridges, 1941 (see Appendix A). Substitution of the foregoing values in Eqs. 24 to 29 gives the following working formulas: For  $l/i = 0$  to  $l/i = 152$ —

$$\left(\frac{P}{A}\right)_w = 15,000 - 0.323 \left(\frac{l}{i}\right)^2 \dots\dots\dots (30a)$$

and, for  $l/i$  larger than 152—

$$\left(\frac{P}{A}\right)_w = \frac{34,000}{1 + \frac{1}{6,510} \left(\frac{l}{i}\right)^2} \dots\dots\dots (30b)$$

Eq. 30a is almost identical with that of the A.R.E.A. Bridge Specifications (Section III) for slenderness ratios as great as 140:

$$\left(\frac{P}{A}\right)_w = 15,000 - \frac{1}{3} \left(\frac{l}{i}\right)^2 \dots\dots\dots (31)$$

For a greater slenderness ratio the A.R.E.A. specifications suggest the direct use of the secant formula. It is thought that the use of the more convenient Eq. 30b, instead of the secant formula, is preferable in a practical way.

The exact secant curve (solid curve), as well as the approximating working curves (dotted curves), is shown in Fig. 1(a). The close approximation obtained is immediately evident.

Example 2.—Consider a column of high-strength, corrosion-resisting steel under the following design conditions:  $\sigma_y = 65,000$  lb per sq in.;  $E = 29,500,000$  lb per sq in.;  $\eta = 1.55$ ;  $a = 0.35$ ; and  $k = 0.75$ . Substitution of these values

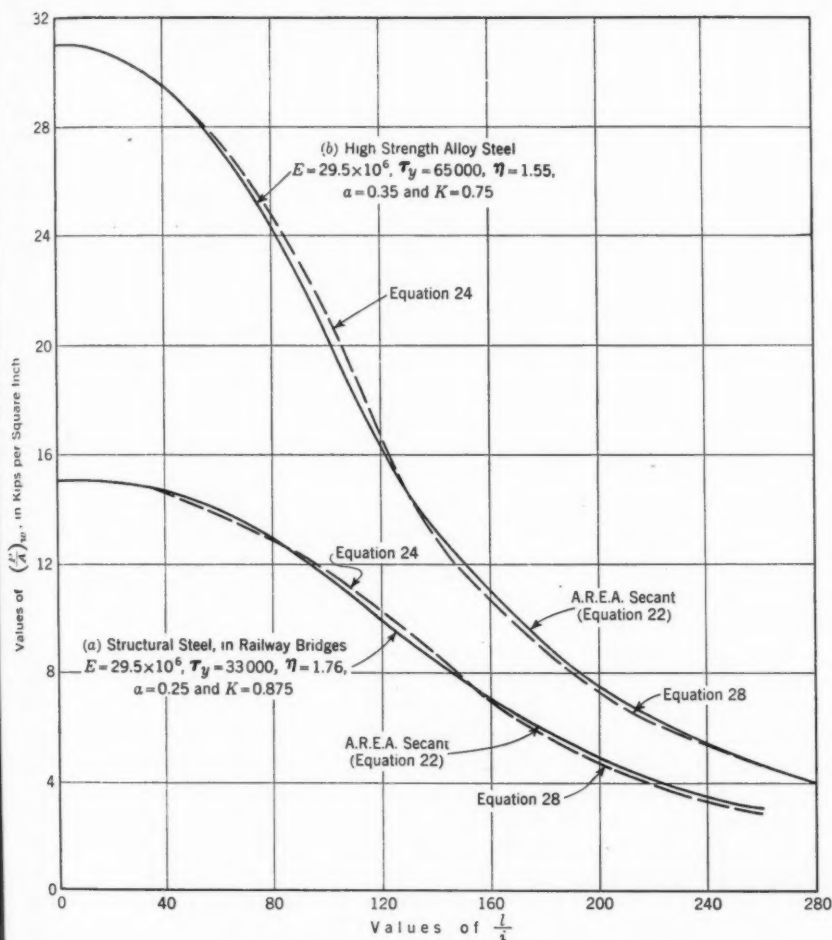


FIG. 1.—DESIGN CURVES FOR STEEL COLUMNS

in Eqs. 24 to 29 gives the following working formulas: For  $l/i = 0$  to  $l/i = 126$ —

$$\left(\frac{P}{A}\right)_w = 31,000 - 0.971 \left(\frac{l}{i}\right)^2 \dots\dots\dots (32a)$$

and, for  $l/i$  larger than 126—

$$\left(\frac{P}{A}\right)_w = \frac{55,700}{1 + \frac{1}{6,140} \left(\frac{l}{i}\right)^2} \dots\dots\dots (32b)$$

The curves for high-strength steel corresponding to Eqs. 32a and 32b and to the secant formula are given in Fig. 1(b).

Examples 1 and 2 demonstrate the adaptability of the writer's proposal, the simple form of the resulting working formulas, and the good degree of approximation as compared with the values of the secant formula.

For each of his nine formulas the author has indicated how they can be made to fit a particular series of tests. The same can be accomplished by the secant formula, as was actually done in the Society's column investigation.<sup>9</sup> Whereas fitting an empirical formula to a series of tests consists in the adjustment of arbitrary coefficients, the rational way to coordinate the secant formula with test results is to determine experimentally the degree of imperfection entering the formula. To establish the magnitude of this quantity it is only necessary to know  $e + d$ ; that is, eccentricity plus crookedness (compare Eq. 23). The effect of these two factors is known to be almost identical; that is, an eccentricity of given amount has the same effect as a deviation from straightness of the same amount. For this reason it is superfluous to determine  $e$  and  $d$  separately, since only their sum enters the secant formula (and Eqs. 24 and 28). This quantity ( $e + d$ ) is easily determined in standard ways by the well-known Southwell method.<sup>10</sup> By this method it is even possible to determine the degree of imperfection without loading the element to destruction, which is of considerable advantage in the case of complicated special structures. Once this quantity, and the physical characteristics of the material are known, the secant formula and the corresponding simplified expressions, Eqs. 24 and 28, can be applied directly.

It is quite possible that one of the author's several formulas can be made to fit the secant equation equally as well as Eqs. 24 and 28. The writer has chosen these particular formulas because they are of a form familiar to designers and widely used in design codes. Experience has shown that there is no particular disadvantage in using two different formulas for different ranges of  $l/i$ . The main point the writer wishes to convey is that, in his opinion, analytically rigorous formulas which lend themselves to complete physical interpretation are vastly preferable to empirical expressions of a purely formal, mathematical character.

A discussion by the author of the full significance of his arbitrary constant  $S$ , referred to as the secant yield strength, would contribute to a better understanding of this paper.

JONATHAN JONES,<sup>11</sup> M. Am. Soc. C. E.<sup>11a</sup>—This paper is of a philosophical nature, comprising a discussion of how various curves, each defined by a formula, can be made to fit the data derived from a large number, and wide variety, of tests made on columns. It is of no assistance to a practicing engineer in what is considered "structural" work, because it proceeds, as expressed by the author in the "Synopsis" and in the "Summary," from postulates which, from the viewpoint of such an engineer, are untrue or unimportant.

The "Synopsis" states that a prime requisite of any column formula is that the average stress "approach the Euler value as the ratio of slenderness becomes

<sup>10</sup> "Theory of Elastic Stability," by S. Timoshenko, 1st Ed., McGraw-Hill Book Co., Inc., New York and London, 1936, p. 177.

<sup>11</sup> Chf. Engr., Fabricated Steel Constr., Bethlehem Steel Co., Bethlehem, Pa.

<sup>11a</sup> Received by the Secretary January 16, 1945.

infinite." As the Euler value at that ratio is zero, and as the column formulas in practical use do approach zero as the ratio of slenderness becomes infinite, this statement seems of little importance. However, it is here implied, and elsewhere stated, that a desirable property of a column formula is that it shall be susceptible to extrapolation. The practicing engineer is not concerned with extrapolation (as to values of slenderness ratio), as his entire work is within a range of slenderness ratios defined by specification.

The "Synopsis" further states that a desirable requisite is that "the first derivative of the average stress with respect to the ratio of slenderness be continuous." This condition certainly does not present itself as a necessity to the practicing engineer. Why is it stated as axiomatic?

It is stated in the "Summary" that "there is no point in using a formula that is applicable only \* \* \* up to a given value of the ratio of slenderness \* \* \*." As a matter of fact, almost all of the critical work done by the structural designer is in a range up to a given value of the ratio of slenderness, and a formula that is applicable within that range is exactly what is needed.

It is stated in the "Summary" that

"There is little to be said for a formula on the grounds of its simplicity, for any formula can be plotted to as large a scale as desired and values read from the graph, or it may be tabulated with increments in the argument as small as desired."

As a matter of fact, a practicing engineer will be confronted frequently with a need for an estimate of column capacity, at a time and place where neither graphs nor tables are available; the writer is certain that he has made such estimates, in the course of his experience, thousands of times, by the simple application of mental arithmetic, either at a saving of effort as compared with looking up a graph or a table, or when neither was at hand. It is very largely for this reason that the practicing engineer is so fond of "straight-line" or "Johnson's parabola" formulas, which lend themselves to quick, mental solution.

The attitude of the structural engineer toward column formulas has undoubtedly been oversimplified; but the correct approach for him to take is entirely different from the path marked out in this paper.

The theory leading to the "secant formula" is as valid a piece of mechanical reasoning as that leading to the customary beam theory, the "theorem of three moments," and other tools of the structural engineer. Furthermore, the secant formula has been shown to conform well to tests, where the tests have been controlled with respect to the variables entering into the formula. The point which the structural engineer must remember, however, is that, in order to use the secant formula, there must be established not only the properties of the material and the radius of gyration of the column but also an effective length (between points of contraflexure) and a constant representing the possible external moment applied through frame action, eccentricity of end loading, and crookedness resulting from fabrication, shipment, or erection. If practical column formulas need to be rendered more precise and, therefore, more complicated, the need is to recognize that these moments vary quite considerably

over the entire column field. They may be safely estimated for one category of columns, and again safely estimated for some other category of columns; but it is probably useless to attempt what has been tried in the past—namely, to estimate them once and for all over the entire field.

It would appear to the writer that the most fruitful field of inquiry now open to the structural engineer is to determine, for each different type of column (compression members in framed trusses, columns in mill buildings, columns in story buildings, derrick masts, etc.), the most reasonable allowances for free length and for applied moments that should be inserted in the general secant formula, to safely and economically meet the probable loading conditions in that particular category of columns.

Having established these several allowances for moment, there can be written down, for steel of any one yield-point strength, a secant formula expressing the average unit stress at that load which causes a yield-point stress on the most severely stressed fiber. For a series of other steels, having other yield points, all these formulas can again be written, and the difference of yield point will make them all different, in detail only, from the corresponding members of the first series.

It is reasonable to believe, from the work already reported by a committee of the Society, that each of these formulas would quite reasonably represent the maximum loads that would be obtained from a series of tests, introducing in each particular case the material and the moments postulated in setting up the formulas.

The structural engineer would then affect each of these formulas by a factor of safety, representing the number of times (usually between 1.50 and 2), by which the anticipated total load could be assumed to be safely multiplied before the extreme fiber stress would reach the yield point in question. These then would be a set of safe-load formulas, all based upon the same reasonable premises, and each safe for practical use. However, one serious practical difficulty would remain—that the secant formula does not lend itself readily to arithmetical solution. It therefore would remain to be seen whether it would be possible to substitute, for the safe-load formulas thus in hand, simpler formulas, easier to handle, and not departing unduly from the secant series.

In the case of compression members in railroad bridge trusses, it is common knowledge that the American Railway Engineering Association (A.R.E.A.) has found such a substitution possible—

$$P = 15,000 - 0.25 \left( \frac{l}{i} \right)^2 \dots\dots\dots (33a)$$

—for values of slenderness ratio up to 140. The American Institute of Steel Construction (A.I.S.C.) has established a similar parabola—

$$P = 17,000 - 0.485 \left( \frac{l}{i} \right)^2 \dots\dots\dots (33b)$$

—for columns in buildings, up to a slenderness ratio of 120. Beyond these slenderness ratios it has been found, by these specification-writing bodies,

that the secant curve and the second-degree parabola depart from each other rapidly. They have accordingly adopted other curves for slenderness ratios greater than those just stated—A.R.E.A. choosing the secant formula itself, and A.I.S.C. choosing the Rankine-Gordon type, as a reasonable equivalent.

In actual bridge and building practice, inasmuch as columns of greater slenderness ratios than those to which the second-degree parabola can be applied either are extremely rare, or are relegated by the specifications to non-critical elements of the structure, it would seem to the writer to be fully justifiable to adopt, for simplicity, for the second portion or end of each curve, a curve of the Euler form—namely, some easily remembered constant number in the numerator and the familiar  $\left(\frac{l}{r}\right)^2$  in the denominator (the author's  $\left(\frac{l}{i}\right)^2$ ); or an inverted parabola, which also can be kept simple in form and easy to solve.

The writer would not wish it to be inferred that he considers any of the foregoing remarks to be applicable outside the field of usual columns of structural steel, in bridges, buildings, and like structures; in that field, however, he feels that the discussion of column formulas should be entirely uninfluenced by the hypotheses and developments of the present paper.

Correction for *Transactions*: In December, 1944, *Proceedings*, page 1571, line following Eq. 21b, change "Eqs. 20" to "Eqs. 21."

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# AMERICAN SOCIETY OF CIVIL ENGINEERS

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## DISCUSSIONS

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### STRESSES IN THE LININGS OF SHIELD-DRIVEN TUNNELS

#### Discussion

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BY NATHAN D. BRODKIN, M. A. DRUCKER, AND SIGVALD JOHANNESSON

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NATHAN D. BRODKIN,<sup>17</sup> ASSOC. M. AM. SOC. C. E.,<sup>17a</sup>—The primary value of this paper is that it presents a rational method of determining the soil reactions for shield-driven tunnels and dispels much of the "fog" hitherto surrounding an analytical treatment of the subject. The high cost of modern, shield-driven, vehicular and rapid transit tunnels makes it of prime importance to attain the utmost economy of design consistent with reasonably assumed loading conditions. The author's proposal for reducing the weight of metal in the linings of such tunnels deserves the careful thought of tunnel builders.

An elastic tunnel ring subjected to a system of active loads will deform at all points in a definite manner; and, consistent with these deformations, soil reactions of definite magnitude and direction will appear in the surrounding material, dependent only on the elastic properties of the soil and the flexibility of the tunnel lining. The customary practice of assuming arbitrarily that the vertical soil reactions are of uniform intensity is basically erroneous and may lead (as indicated by the author) to uneconomical designs.

Selecting the invert joint as a point of reference, the author has derived expressions for  $A$ -constants (Eqs. 8), which, when multiplied by the external radial or tangential forces, yield the moment, thrust, and shear at that joint, provided the system is in static equilibrium. When referred to the developed periphery of the ring, the  $A$ -constants represent the ordinates to the influence lines for moment, thrust, and shear at the invert joint due to a unit load acting radially or tangentially at sixteen points on the circumference of a ring of unit radius.

For example, the constants  $A_{Mr}$  and  $A_{Mt}$ , when so plotted, yield the influence diagrams shown in Fig. 17. To locate the maxima and minima of the

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NOTE.—This paper by Anders Bull, M. Am. Soc. C. E., was published in November, 1944, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: January, 1945, by J. A. Van den Broek.

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<sup>17a</sup> Received by the Secretary December 20, 1944.

influence curves, first derivatives of the expressions in Eqs. 8 are equated to zero, yielding the angular values of these points. The influence ordinates corresponding to these values may then be found by resubstitution in Eqs. 8.

In Table 1 the  $A$ -constants have been determined for intervals of  $22^\circ 30'$ . Where force concentrations are applied anywhere between the sixteen tabu-

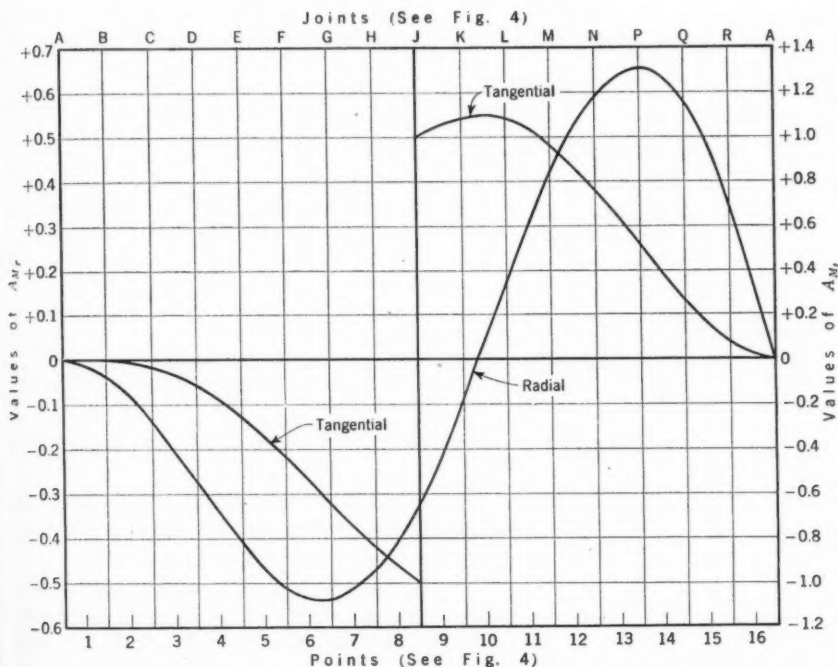


FIG. 17.—MOMENT INFLUENCE LINE FOR RADIAL FORCES AND TANGENTIAL FORCES

lated points, a linear interpolation of the  $A$ -values might lead to inaccurate results. Influence diagrams plotted to a large scale give a truer indication of point-to-point variation, permit the intermediate values to be determined more accurately, and furnish a means for determining graphically the reactions for any system of loads in static equilibrium. Since any joint may be assumed as a point of origin similar to the invert joint, the same influence diagrams may be used for any joint by shifting the joint and force notation—in a manner similar to the analytical procedure described by the author in Section 11.

Eqs. 8 are basic for any ring-shaped structure, greatly simplifying the mathematical work of deriving equations for moment, thrust, and shear in tunnel rings due to any given loading, as an example will serve to illustrate.<sup>18</sup>

The ring, shown in Fig. 18, is subjected to two equal and opposite horizontal forces  $P$  concentrated at  $\theta$  and  $2\pi - \theta$ . An equation must be written for moment at point A.

<sup>18</sup>"Stress Coefficients for Large Horizontal Pipes," by James M. Paris, *Engineering News-Record*, November 10, 1921, p. 768.

In Fig. 18, the horizontal forces  $P$  have been replaced by their radial and tangential components,  $P \sin \theta$  and  $P \cos \theta$ , respectively. Substituting in Eq. 6c for  $A_{Mr}$  and  $A_{Mt}$  from Eqs. 8 and replacing  $\alpha$  by  $\theta$  and  $2\pi - \theta$ , respectively, the separate effects of radial and tangential forces may be written as follows: Moment produced by radial forces—

$$M'_A = -\frac{Pr}{2\pi} \sin \theta (\theta \sin \theta + 1 - \cos \theta) - \frac{Pr}{2\pi} \sin \theta [- (2\pi - \theta) \sin \theta + 1 - \cos \theta] = -\frac{Pr}{\pi} [-(\pi - \theta) \sin^2 \theta + \sin \theta - \sin \theta \cos \theta] \dots (30a)$$

and moments produced by tangential forces—

$$M''_A = +\frac{Pr}{2\pi} \cos \theta \times \theta (1 - \cos \theta) - \frac{Pr}{2\pi} \cos \theta (2\pi - \theta) (1 - \cos \theta) = -\frac{Pr}{\pi} [(\pi - \theta) \cos \theta - (\pi - \theta) \cos^2 \theta] \dots \dots \dots (30b)$$

Adding the separate effects:

$$M_A = M'_A + M''_A = -\frac{Pr}{\pi} [-(\pi - \theta) (\sin^2 \theta + \cos^2 \theta) + (\pi - \theta) \cos \theta + \sin \theta - \sin \theta \cos \theta] = +\frac{Pr}{\pi} (\pi - \theta - \sin \theta) (1 - \cos \theta) \dots (31)$$

The writer would like to call attention to the effect caused by a change in the value of the soil compression constant  $K$ . For this purpose the moments produced in the tunnel lining shown in Fig. 11(a) for successive values of  $K$  from  $K = 3$  to  $K = 15$  were computed using the author's method of analysis. The results are summarized in Table 15, and moment diagrams corresponding to the various values of  $K$  have been plotted against the developed ring circumference in Fig. 19. Values for  $K = 12$  in Fig. 19 are taken from Table 14.

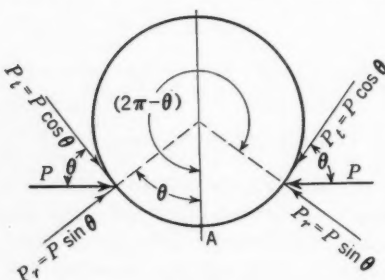


FIG. 18

Fig. 19 shows the progressive reduction in crown moment with an increase in the value of  $K$ . If this moment is plotted against the soil compression constant as in Fig. 20, it is seen that the variation in this reduction is nonlinear. Mr. Drucker,<sup>3</sup> although treating the problem of soil reactions from a different standpoint, obtains a curve of similar nature.

Assuming, as a basis, a sandy soil having a  $K$ -value of 12 to which the example given by the author applies, it is seen from Table 15 that, for decreases in  $K$  to 9, 6, and 3, the crown moment increases 9.5%, 23.7%, and 48.5%, respectively. Similarly, the increase in crown tensile fiber stress is 12.7%,

<sup>3</sup>"Determination of Lateral Passive Soil Pressure and Its Effect on Tunnel Stresses," by M. A. Drucker, *Journal of the Franklin Institute*, May, 1943, p. 499.

32.3%, and 65% for the respective values of  $K$ . When it is considered that a soil having a  $K$ -value as low as 3 might be somewhat plastic, and would exert a greater active lateral pressure than that assumed herein, the percentages

TABLE 15.—EFFECTS OF THE VARIATION OF THE SOIL COMPRESSION CONSTANT ON THE TUNNEL LINING

Description (see Fig. 4)	SOIL COMPRESSION CONSTANTS, $K$									
	3		6		9		12		15	
	$F$	$M$	$F$	$M$	$F$	$M$	$F$	$M$	$F$	$M$
(a) PHYSICAL PROPERTIES ( $I = 136 \text{ In.}^4$ ; $S = 23.4 \text{ In.}^2$ ; AND $A = 27.6 \text{ Sq. In.}$ )										
(b) SOIL REACTIONS, IN KIPS, AND MOMENTS, IN INCH-KIPS										
Joint A	5.941	-14.0	5.986	-1.3	6.071	+2.4	6.154	+5.3	6.227	+6.0
Point 1		-22.3		-11.7		-7.0		-4.5		-3.0
Joint B	5.205	-28.9	5.179	-25.8	5.164	-22.9	5.154	-20.5	5.149	-18.3
Point 2	3.937	-10.6	3.964	-10.7	3.920	-14.1	3.863	-15.3	3.807	-15.8
Joint C	2.239	+57.9	2.548	+39.5	2.680	+29.2	2.747	+22.7	2.783	+17.7
Point 4	....		0.499		0.871		1.154		1.382	
Joint E		+76.6		+64.6		+56.8		+51.2		+46.9
Point 5		+21.2		+23.3		+24.0		+24.4		+24.5
Joint F		-53.3		-42.1		-35.7		-31.7		-28.6
Point 6		-86.4		-72.0		-63.7		-58.2		-54.2
Joint G										
Point 7										
Joint H										
Point 8										
Joint I										
Point 9										
Joint J										
(c) SETTLEMENT $\delta$ , IN INCHES										
....	0.594		0.301		0.204		0.156		0.127	
(d) THRUST, IN KIPS										
Joint F	+37.4		+37.5		+37.6		+37.7		+37.8	
Joint J	+34.7		+35.1		+35.3		+35.5		+35.6	
(e) MAXIMUM STRESS AT JOINT F, IN KIPS PER SQUARE INCH										
$\sigma_b$	76.6/23.4 = -3.27	64.6/23.4 = -2.76	56.8/23.4 = -2.43	51.2/23.4 = -2.19	46.9/23.4 = -2.00					
$\sigma_t$	37.4/27.6 = -1.36	37.5/27.6 = -1.36	37.6/27.6 = -1.36	37.7/27.6 = -1.37	37.8/27.6 = -1.37					
Total	-4.63	-4.12	-3.79	-3.56	-3.37					
(f) MAXIMUM STRESS AT JOINT J, IN KIPS PER SQUARE INCH										
$\sigma_b$	86.4/23.4 = +3.69	72.0/23.4 = +3.08	63.7/23.4 = +2.72	58.2/23.4 = +2.49	54.2/23.4 = +2.32					
$\sigma_t$	34.7/27.6 = -1.26	35.1/27.6 = -1.27	35.3/27.6 = -1.28	35.5/27.6 = -1.29	35.6/27.6 = -1.29					
$\sigma_a$	19.1/27.6 = +0.69	19.1/27.6 = +0.69	19.1/27.6 = +0.69	19.1/27.6 = +0.69	19.1/27.6 = +0.69					
Total	+3.12	+2.50	+2.13	+1.89	+1.72					

for  $K = 3$  are perhaps too high. Unless conclusive evidence is at hand as to the nature of the soil, it would appear prudent to assume a conservative value for  $K$ .

From the foregoing, it is evident that increasing the soil compression constant has the same effect as increasing the flexibility of the lining. The simple

relationship existing between  $I$  and  $K$  may be shown in the following manner: Eq. 18 may be written in the form,

$$\Sigma(F D_1) - \Sigma(P D_1) = + \frac{E I}{r^3} \cos \alpha_1 \delta - \frac{E I}{r^3 K A'} F_1 \dots \dots (32a)$$

or

$$\frac{[\Sigma(F D_1) - \Sigma(P D_1)] r^3}{E} = I \cos \alpha_1 \delta - \frac{I F_1}{K A'} \dots \dots \dots (32b)$$

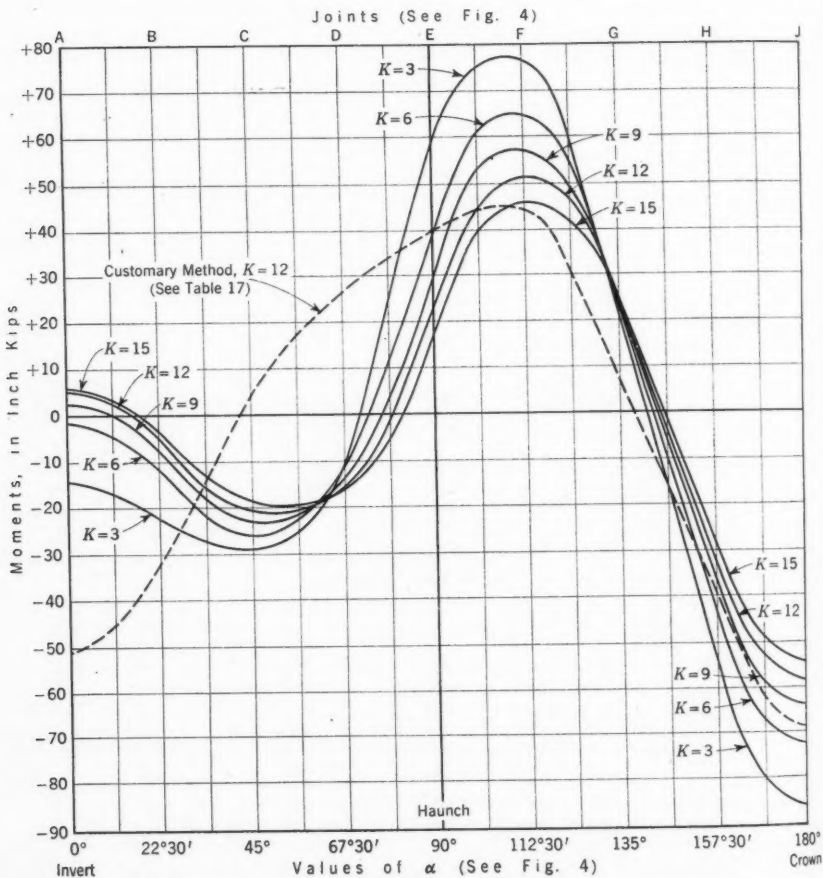


FIG. 19.—MOMENTS IN THE TUNNEL LINING RELATED TO THE SOIL COMPRESSION CONSTANT ( $I = 136 \text{ In.}^4$ )

Denoting by  $F_A$  the soil reaction on a segment extending from point 1 to point 16, Eq. 15 takes the form  $F_A = \delta K A'$  or  $\delta = \frac{F_A}{K A'}$ . Substituting, therefore, for  $\delta$  in Eq. 32b:

$$\begin{aligned} \frac{[\Sigma(F D_1) - \Sigma(P D_1)] r^3}{E} &= I \cos \alpha_1 \frac{F_A}{K A'} - \frac{I F_1}{K A'} \\ &= \frac{I}{K A'} (F_A \cos \alpha_1 - F_1) \dots \dots \dots (33a) \end{aligned}$$

or

$$\frac{I}{K} = \frac{[\Sigma(F D_1) - \Sigma(P D_1)] r^3 A'}{E (F_A \cos \alpha_1 - F_1)} \quad (33b)$$

Since there is a definite relationship between the soil reactions  $F_1, F_2, \dots, F_5$  for any given loading,  $F_A$  must likewise bear a fixed relationship to  $F_1$ .

From Eq. 33b, it is seen that, if the value of  $K$  is multiplied by a certain factor, it has the same effect as if the value of  $I$  were divided by the same factor.

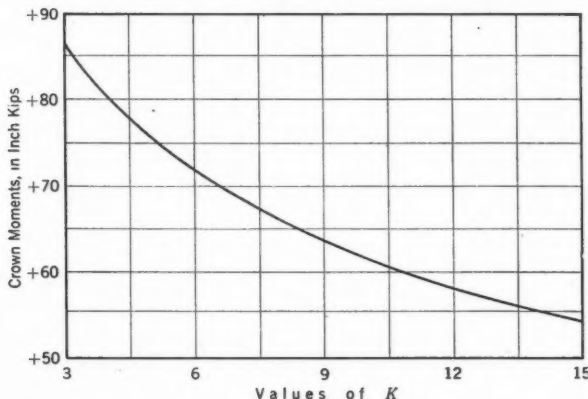


FIG. 20.—CROWN MOMENT IN THE TUNNEL LINING RELATED TO THE SOIL COMPRESSION CONSTANT ( $I = 136 \text{ in.}^4$ )

Stated in other words, if the soil compression constant is increased, it has the same effect as a corresponding increase in the flexibility of the lining.

It is interesting to note in Table 15 that, for the various values of  $K$  for which analyses were made, the reactive forces  $F$  vary only slightly at similar points on the ring for different values of  $K$ . These small variations, however, are sufficient to cause large changes in the moments.

It is further noted in Table 15 that, as the soil compression constant is decreased, the effects of soil reaction diminish on the upper parts of the tunnel. For low values of  $K$  and a moment of inertia of  $136 \text{ in.}^4$ , the reactive forces are confined to the lower half of the tunnel periphery.

Contrary to the usual conception of a flattening at the invert of the ring with a predominance of downward loads, the results obtained using the author's method of analysis indicate that it is quite possible for the curvature to sharpen at the invert. The computed radial deformations of the lining due to combined active loads and soil reactions, corresponding to the assumed values of  $K$ , have been listed in Table 16. The final displacements are due to three factors—active loads, soil reactions, and the settlement. Actual deformation of the ring is a function of the first two factors only. A negative radial deformation means that the point has moved inward and a positive value indicates an outward movement with respect to the position of the ring before it is loaded. Table 16 shows quite clearly that, for high values of  $K$ , points 1 and 2 move inward with respect to joint A. As the soil compression constant is

decreased, the soil, becoming more yielding, allows the lining to flatten at points near the invert under the effects of active loads, until a value of  $K$  is reached where all points from 1 to 6, inclusive, move outward. This is the case for an assumed value of  $K = 3$ . Thus, for high values of  $K$  it is possible to obtain moments at the invert causing tension on the outer fibers.

TABLE 16.—DEFORMATION, IN INCHES, SHOWING THE EFFECT OF A VARIATION IN THE SOIL COMPRESSION CONSTANT ON THE RADIAL DEFORMATION OF THE LINING ( $I = 136$  IN.<sup>4</sup>)

Point	$K = 3$	$K = 6$	$K = 9$	$K = 12^a$	$K = 15$
1	+0.0002	-0.0014	-0.0020	-0.0022	-0.0023
2	+0.0168	+0.0041	-0.0011	-0.0033	-0.0045
3	+0.0562	+0.0273	+0.0147	+0.0080	+0.0041
4	+0.1036	+0.0663	+0.0477	+0.0369	+0.0297
5	+0.1092	+0.0830	+0.0682	+0.0588	+0.0517
6	+0.0275	+0.0325	+0.0331	+0.0327	+0.0320
7	-0.1084	-0.0687	-0.0428	-0.0381	-0.0308
8	-0.2114	-0.1496	-0.1187	-0.0994	-0.0867

<sup>a</sup> See line 9, Table 12.

With an increase in the soil compression constant, the soil reactions tend to reduce the moments in the lower half of the lining, and, with a simultaneous reduction in the amount of the displacement at the horizontal diameter as shown in Table 16, they cause this part of the ring to act somewhat as an abutment for the upper half. The upper part of the ring then behaves much as an arch subject mainly to the effects of active loads.

For purposes of comparison, the writer has analyzed the tunnel chosen by the author as an example in Section 21 (for which  $I = 136$  in.<sup>4</sup>) by the customary method of analysis described in Section 3, assuming (1) the vertical soil reactions to be distributed uniformly over the horizontal diameter (Fig. 1, loading 12); and (2) an induced lateral passive pressure of an intensity dependent on the increase in the horizontal diameter under active loads. The effects

TABLE 17.—COMPARISON OF MOMENTS, IN INCH-KIPS ( $K = 12$  AND  $I = 136$  IN.<sup>4</sup>)

Moments	JOINT (SEE FIG. 4)								
	A	B	C	D	E	F	G	H	J
Customary method—									
Caused by active forces...	-230.2	-166.5	-7.6	+161.1	+242.2	+179.0	+4.5	-173.8	-247.9
Caused by passive forces...	+179.2	+134.6	+11.8	-134.8	-202.5	-134.8	+11.8	+134.6	+179.2
Net moment.....	-51.0	-31.9	+4.2	+26.3	+39.7	+44.2	+16.3	-39.2	-68.7
Author's method (net moment)	+5.3	-4.5	-20.5	-15.3	+22.7	+51.2	+24.4	-31.7	-58.2

of lateral passive pressure are in accordance with the pressure diagram shown in Fig. 1, loading 13a, and are based on the theory developed by Mr. Drucker,<sup>2</sup> using a value of  $K = 12$ . The results are shown in Table 17 and the corresponding moment diagram has been plotted in Fig. 19. The moments as determined by the author are also given in Table 17 for purposes of comparison.

Comparing the two sets of values, the difference in results is apparent at once, especially in the lower parts of the lining. The customary analysis yields a crown moment 18% greater than that obtained by the author. The invert moment is about 74% of the crown moment obtained by the customary method, whereas the author's analysis shows a very small moment at this point amounting to about 9% of that at the crown. From the foregoing, it is evident that the customary assumptions relative to soil reactions result in moments in the lining varying considerably from those obtained by a more rational method.

M. A. DRUCKER,<sup>19</sup> Esq.<sup>19a</sup>—A fundamental method of complete stress analysis for tunnels is presented in this paper including the effects of both the active loading and the passive soil resistance, or, as the author designates the latter, "soil reactions." It should be of interest to ascertain the effects of the active loading separately in order to study the effect, on the final stresses, of considering the passive pressures. This would also show to what extent reliance is placed on the capability of the passive pressures to keep the stresses in the tunnel from exceeding the maximum permissible stress. In the author's analysis, however, the two sets of stresses are inseparably combined. Therefore, to form a clear idea of the effect of the proposed method, as compared with what may be termed the ordinary method, it would be necessary first to obtain the stresses due to the active loading according to general practice in tunnel design.

For the purpose of comparison, of course, the active loading and pressures used by the author will have to be followed. However, attention may be called to what appear to be several discrepancies. In considering the effect of the weight of the tunnel ring, 220 lb per sq ft is used. This weight consists of 110 lb for the weight of the cast-iron ring having a sectional area of 27.6 sq in. and of 110 lb for the weight of the concrete lining; but, as the concrete lining is poured in normal air, a considerable time after the cast-iron ring is erected, the weight of the cast iron alone should have been considered in conjunction with the effect of the internal air pressure used to keep out the water while erecting the cast-iron lining. In obtaining the moment of inertia and section modulus of the tunnel ring, the author correctly disregards the concrete lining.

Another point that may be noted is that the author apparently neglects the buoyant effect of the water on the soil when obtaining the lateral active pressures against the tunnel. Using Eq. 22c he takes a unit active horizontal pressure equal to one third the weight of overlying soil—100 lb per cu ft. This corresponds to an active pressure due to a dry soil having an angle of repose of 30°. If the buoyant effect of the water were considered, the unit active lateral earth pressure, of course, would be less than 33.3 lb. The variation in stress resulting from this difference in pressure will be considered subsequently.

Tunnel design analysis, for active loading, based on the ordinary method, resulted in a simplification by employing coefficients for bending moments and

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<sup>19a</sup> Received by the Secretary December 21, 1944.

thrusts for the component parts of the total loading and pressures. These coefficients, for the stresses at point J, are given in Table 18, Cols. 2 and 4. In the problem considered by the author (see Section 21), the outside diameter of the tunnel is 17 ft 5 in. and the radius at the neutral axis is 8.5 ft. Loaded on the tunnel section are an earth cover of 36 ft and 27 ft of water. The net

TABLE 18.—COMPUTATION OF STRESSES FOR POINT J (SEE FIG. 4) (BASED ON INDICATED COEFFICIENTS FOR BENDING MOMENT AND THRUST)

Load- ing No. <sup>a</sup>	Loading (1)	MOMENTS, IN FOOT-POUNDS		THRUSTS, IN POUNDS	
		Formula (2)	Moment (3)	Formula (4)	Thrust (5)
1	$w_1 = 36 \times 100 = 3,600$	$-0.25 w_1 r^2$	-65,000	0	0
2	$w_2 = 100$	$-0.027 w_2 r^2$	-1,650	$-0.021 w_2 r^2$	-152
3	$w_3 = 27 \times 25 = 675$	$-0.25 w_3 r^2$	-12,300	0	0
4	$w_4 = 25$	$-0.027 w_4 r^2$	-410	$-0.021 w_4 r^2$	-38
5	$w_5 = 220$	$-0.345 w_5 r^2$	-5,480	$-0.167 w_5 r$	-312
7	$w_7 = 63$	$-0.009 w_7 r^2$	-360	$-0.021 w_7 r^2$	-96
8	$w_8 = \frac{1}{2} \times 3,600 = 1,200$	$+0.25 w_8 r^2$	+21,650	$+1.00 w_8 r$	+10,200
9	$w_9 = 33.3$	$+0.208 w_9 r^2$	+4,250	$+0.625 w_9 r^2$	+1,510
10	$w_{10} = 27 \times 63 = 1,700$	$+0.25 w_{10} r^2$	+30,700	$+1.00 w_{10} r$	+14,400
11	$w_{11} = 63$	$+0.208 w_{11} r^2$	+8,030	$+0.625 w_{11} r^2$	+2,850
....	Summation	....	-20,570	....	+28,362

<sup>a</sup> The loading numbers correspond to those shown in Fig. 1; loading 6 is omitted because its effect is included in loadings 1 to 5.

sectional area of the ring is 27.6 sq in., its gross moment of inertia is 136 in.<sup>4</sup>, and the intrados section modulus is 23.4 in.<sup>3</sup>, all per foot of tunnel length. Applying the coefficients of Table 18 to this case, the stress at point J is  $20.57 \times \frac{12}{23.4} - \frac{28.36}{27.6} = 10.53 - 1.03 = 9.50$  kips per sq in. The author finds the final stress at this point to be +1.89 kips per sq in., including +0.69 kip per sq in. due to the air pressure, which is independent of the effect of the external loading. Therefore, the remaining stress, without air pressure, would be +1.2 kips per sq in., so that, in accordance with the method presented in the paper, the tensile stress reduction due to the passive pressures may be said to be 9.5 - 1.2 or 8.3 kips per sq in.

On the loading diagram, Fig. 1, the author represents by loading 13a the distribution of passive pressures as considered by the writer.<sup>3</sup> He states (see Section 3) that these pressures are "continuous functions of the depth." It would be more correct to describe this method as one in which the horizontal passive pressures are assumed proportional to the lateral deflections caused by the active loading. In conjunction with this method, the excess of the downward vertical loading over the upward water pressure is assumed to be uniformly distributed below the horizontal diameter in accordance with general practice. The author states several times that such assumed vertical pressure distribution is the cause of high calculated stresses. From the following

computations it should become clear that such pressure distribution does not result in large computed stresses.

To determine the stress reduction due to the passive pressures in accordance with the writer's method, the horizontal deflection at the horizontal tunnel axis due to all the active loading and pressures will first be found from the resulting bending moment at point J—the assumption being that the deflections due to all the active loadings will be the same as the deflection due to a uniform vertical loading causing the same moment at this point.

For a uniform vertical loading of  $w$  lb per sq ft,  $M_J = 0.25 w r^2$ . Table 18 gives  $M_J$  as 20,570 ft-lb. Therefore,  $0.25 w r^2 = 20,570$  and  $w = \frac{20,570}{0.25 (8.5)^2} = 1,140$  lb per sq ft. For such uniform load the deflection  $d_1$  at the horizontal diameter is

$$d_1 = \frac{144 w r^4}{E I} \dots \dots \dots (34)$$

For the problem under consideration,  $d_1 = \frac{144 \times 1,140 \times (8.5)^4}{12,000,000 \times 136} = 0.526$  in.

The effect of the passive pressures may be found in the following manner:  $p = 1.28 w = 1.28 \times 1,140 = 1,460$  lb. In this computation  $p$  is the maximum intensity of a pressure, varying as  $\sin^2 \alpha_J$  (see Fig. 21), that would be capable of deflecting the tunnel back to its original position. This is an extreme pressure and is obtained only as a necessary intermediate step.

With the foregoing information and the given value of 12,000 lb for the soil constant,  $K$ , the reduction in stress,  $\sigma_x$ , at point J, due to the passive pressures may be found from the following formula derived by the writer:

$$\sigma_x = \frac{12 \times 0.182 K r^2 p d_1}{\frac{I}{c} (K d_1 + p)} + \frac{0.59 K r p d_1}{A (K d_1 + p)} \dots \dots \dots (35)$$

Substituting the proper values in Eq. 35,

$$\sigma_x = \frac{2.18 \times 12,000 \times (8.5)^2 \times 1,460 \times 0.526}{23.4 (12,000 \times 0.526 + 1,460)} + \frac{0.59 \times 12,000 \times 8.5 \times 1,460 \times 0.526}{27.6 (12,000 \times 0.526 + 1,460)} = 8.22 \text{ kips per sq in.}$$

As previously indicated, the author's method gives a stress at point J of 8.3 kips per sq in. less than that obtained by the ordinary method for active loading only. Comparing this value with the foregoing stress reduction of 8.22 kips per sq in. due to the passive pressures, the difference in the results obtained by

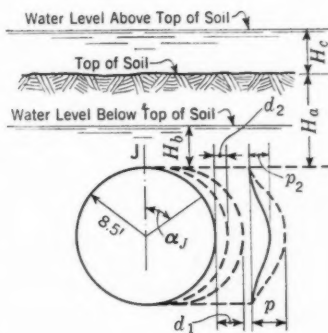


FIG. 21

the two methods is found to be only about 1%. It is gratifying, indeed, to find that the writer's method, which includes approximations for the sake of simplicity and which requires only simple slide-rule computations, gives results in such close agreement with the author's more extensive analysis.

With the reduction in stress of 8.22 kips per sq in., due to the developed passive pressures, the stress at point J would become  $9.5 - 8.22 = 1.28$  kips per sq in. without considering the stress due to air pressure, and  $1.28 + 0.69 = 1.97$  kips per sq in. considering the stress due to air pressure, as against the author's 1.2 kips per sq in. and 1.89 kips per sq in., respectively.

At times it may be more desirable to determine the value of the soil constant,  $K$ , necessary to keep the stress from exceeding the maximum allowable value, than to begin with an assumed value for this constant to find the resulting stress. In the present stage of knowledge of soil mechanics, it would be rather difficult to defend any definite  $K$ -value even if it were based on the best possible available information as to the nature of the soil through which the tunnel is to be constructed. Surely, one could not contend for a value of  $K = 12$  while another argued for a value of, say, 10 or 8.

Knowing the stress at point J due to the active loading, computed as indicated in Table 18, and the maximum permissible tensile stress, the necessary value of the soil constant may be obtained from the following formula derived by the writer:

$$K = \frac{\sigma_x}{d_1 \left[ \frac{2.18 r^2}{I} + \frac{0.59 r}{A} - \frac{\sigma_x}{p} \right]} \dots \dots \dots (36)$$

in which, as before,  $\sigma_x$  is the necessary stress reduction to be provided by the passive pressures.

For a maximum allowable tensile stress of 4 kips per sq in. for cast iron and taking into account the stress due to air pressure,  $\sigma_x$ , for the problem under discussion, would have to be  $9.50 + 0.69 - 4.00 = 6.19$  kips per sq in. Substituting this and other known values in Eq. 36,

$$K = \frac{6.19}{0.526 \left( \frac{2.18 (8.5)^2}{23.4} + \frac{0.59 \times 8.5}{27.6} - \frac{6,190}{1,460} \right)} = 4.37.$$

This result shows that, for the tunnel section considered, there would be no reason for urging a soil constant of  $K = 12$  in checking as to whether the tunnel ring is of sufficient strength for a certain location, when one of only  $K = 4.4$  would keep the stress from exceeding 4 kips per sq in.

In this discussion, reference has been made to the stresses at point J only, because it is the point of greatest tensile stress and, therefore, governs for cast iron. Theoretical computations, including those by the author, as well as tests on rings used for the London tubes,<sup>20</sup> show this to be the case. The loading during these tests (which were conducted under conditions approxi-

<sup>20</sup> "Tunnel Linings with Special Reference to a New Form of Reinforced Concrete Lining," by G. L. Groves, *Journal, Institution of Civ. Engrs.*, London, England, March, 1943, p. 29.

inating those affecting a tunnel after construction) was increased until cracks appeared in the rings—at the crown of the tunnel. Also, if the stress is known at one point, the stress at any other point can be determined readily, if for any unusual reason it should be desired to do so, by considering the bending and thrust at the point of known stress in conjunction with the known or assumed loading, including the passive pressures.

As previously noted, the author assumed the weight of the tunnel ring as 220 lb per sq ft whereas he should have used only 110 lb for the ring having a sectional area of 27.6 sq in. He also neglected to consider the buoyant effect of the water on the surrounding soil. Making the correction for the weight of the tunnel ring, the final stress at J for this ring was found to be 1.71 kips per sq in. as indicated in Table 19 for condition C. Correcting for the effect of

TABLE 19.—COMPARISON OF STRESSES AT POINT J

Con- dition	Description	STRESS <sup>a</sup> $\sigma_J$	
		A = 27.6	A = 11.0
A	Author's specified conditions (see Section 21) and method	1.89	3.18
B	Conditions A, with passive pressures based on the writer's method	1.97	3.70
C	Conditions B, with corrected tunnel weights <sup>b</sup>	1.71	2.95
D	Conditions C, taking buoyant effect of water into account, with angle of repose = 30°	2.71	4.90
E	Conditions D, with angle of repose = 45°	3.87	7.50
F	Conditions C, except that all active side earth pressure is disregarded	5.10	9.40

<sup>a</sup>Stresses are in kips per square inch, for section areas of 27.6 sq in. and 11.0 sq in., respectively. <sup>b</sup>For A = 27.6 sq in., the weight was corrected to 110 lb; and, for A = 11.0 sq in., the weight was corrected to 50 lb.

buoyancy, assuming 40% voids, and taking the angle of repose of the soil as 30° gave the effective weight of the soil below water as  $100 - 0.60 \times 63 = 62.2$  lb per cu ft and the unit active earth pressure as  $62.2 \times \frac{1 - \sin 30^\circ}{1 + \sin 30^\circ} = 62.2 \times \frac{1 - 0.5}{1 + 0.5} = 20.7$  lb. For this reduced active earth pressure below water the stress at J was found to be 2.71 kips per sq in. for condition D.

The angle of repose was taken to be 30° as assumed by the author. It may not be out of place, however, to give this assumption a little thought. An angle of repose of 30° is considered a reasonable basis for obtaining side pressures on structures, whose safety requires that the highest possible active pressure be assumed to provide sufficient resistance against it. In the case of tunnels, however, as the active side earth pressures reduce the tensile stress at the governing point, or at the crown of the tunnel, it becomes necessary to decide on an angle of repose that would give active earth pressures which may be counted on to exist at all times. This may be a rather difficult question to decide; but the angle of repose for maximum active earth pressure should not be used when the minimum pressure is desired. In this connection, it may be noted that Lieutenant-Commander Spangler<sup>7</sup> assumes no active side earth

<sup>7</sup>"The Structural Design of Flexible Pipe Culverts," by M. G. Spangler, *Bulletin No. 153*, Eng. Experiment Station, Iowa State College, Ames, 1941.

pressure whatever although he analyzes fills at culverts. Therefore, for such a soil as that to which the author assigns a value of  $K = 12$  for the soil constant, which is compact and undisturbed, an angle of repose of  $45^\circ$  would not be too large. For such angle of repose, and taking buoyancy into account, the unit active earth pressure would be  $62.2 \times \frac{1 - \sin 45^\circ}{1 + \sin 45^\circ} = 10.7$  lb. Based on this active earth pressure the stress at J was found to be 3.87 kips per sq in. for the heavier tunnel section as indicated for condition E, Table 19. Neglecting all active side earth pressure,  $\sigma_J$  was found to be + 5.10 kips per sq in. for condition F. As the angle of repose of the undisturbed soil would very likely be greater than  $45^\circ$ , the final stress, even for so large a soil constant as  $K = 12$ , would be between + 3.87 and + 5.10 kips per sq in.

The low stresses determined by Mr. Bull prompted him to state (see Section 27) " \* \* that with a lining as actually designed more than 70% of the cast iron is wasted." That these low stresses are not inherent in Mr. Bull's method of analysis is demonstrated in Table 19 which shows that, for the same tunnel ring, the same low stresses were obtained by the writer's method under the conditions assumed by the author. The low stresses were due to two factors: (1) The author neglected the buoyant effect of the water on the soil when computing the active horizontal soil pressures; and (2) he assumed a high value for the soil constant without assuming a correspondingly large angle of repose. A more correct basis is that represented by condition E which includes the effects of buoyancy and assumes an angle of repose of  $45^\circ$ . For this condition  $\sigma_J$  was found to be 3.87 kips per sq in. which is close enough, for all practical purposes, to the limiting design stress of 4 kips per sq in. for cast iron.

To compare the writer's method with that of the author for lighter ring sections, a ring having a moment of inertia of 8.5 in.<sup>4</sup>, or one sixteenth of that so far considered, was analyzed. For condition B, Table 19, the stress at J was found to be 3.70 kips per sq in. as against 3.18 kips per sq in. given by the author. The difference between the two stresses should not be surprising in view of the fact that, due to the active loading, the stress for this light ring was found to be 66.0 kips per sq in. so that the stress of 3.70 kips per sq in. represents a reduction in stress of 62.3 kips per sq in. due to the passive pressures. For the other conditions, after making corrections for the weight of the ring, the stresses increase somewhat more than for the heavier section (see Table 19).

For the tunnel ring having an area of 27.6 sq in. the stress at J, computed by the writer's method, was found to be 0.08 kip higher than that given by the author. For the ring having an area of 11.0 sq in. the stress obtained by the writer is about 0.5 kip higher than that given by the author. However, for the area of 17.4 sq in. the writer's value for the stress was found to be 0.9 kip less than that given by the author. In an effort to interpret these varying and irregular differences both sets of stresses were plotted in Fig. 22. As may be seen the stresses computed by the writer (identified by the letter D in Fig. 22) could be joined by a fairly smooth curve marked  $L_D$  whereas the

author's stress points, due to the stress of 3.14 kips for the area of 17.4 sq in., cannot be so joined except for the stress points for 11.0 sq in. and less, through which a curve  $L_B$  could be passed. The irregularity of the other points may be explained by the fact that, as indicated in Table 14, for areas of 11.0 sq in. and less the stresses were obtained on the basis of having six ring segments in contact with the soil whereas for heavier sections only five segments on each side of the tunnel were considered pressing against the soil.

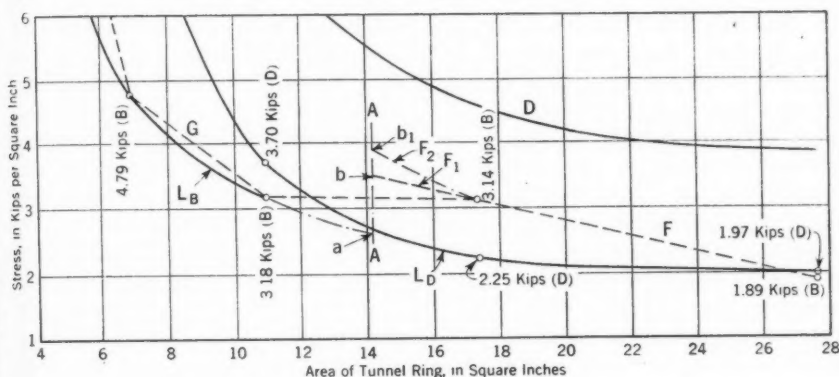


FIG. 22

Assuming that all stress values given by the author are correct, in accordance with his method, the question arises as to what the stress might be for areas between 11.0 and 17.4 sq in. Between these areas there should be one for which, if six segments be assumed in contact with the soil, the pressure on the sixth segment would prove to be zero or nearly so. For such sectional area, which may occur about halfway between those for which the stresses are given, and which may be represented by line A-A in Fig. 22, five segments in contact with the soil could then be assumed. It would seem reasonable that by extending line  $L_B$  toward the right the stress for this area would be at point a or not far from it. Similarly, by extending line F toward the left, the stress for this area of ring would be represented by point b or  $b_1$  if the stress line be assumed to curve upward as suggested by the other stress curves. Although the computed stress values for the area at section A-A may be somewhat different from those indicated in Fig. 22, it would appear that for the same sectional area and loading a consideration of six segments and then five segments would give stresses differing by more than a kip for an average of two stresses of about 3.0 kips. A break in stress values such as at line A-A suggests that the accuracy of computed stress values, by the author's method, for areas on either side of this line would also be questionable. If this be so it would indicate that an accurate method of tunnel design, including the effects of passive pressures, has yet to be developed.

It should also be noted that the stresses represented by lines  $L_B$ ,  $L_D$ , and F are based on the author's loading assumptions; line D shows the stresses that could be expected based on the writer's method, as obtained for condition D,

Table 19, for the assumed loading and active soil pressures which, in his opinion, should have been used.

Mr. Bull asserts that the large critical moments obtained by methods other than his occur because (see "Conclusions") " \* \* \* the erroneous assumption is currently made that the vertical soil reactions are uniformly distributed over the horizontal diameter \* \* \* ." That this is not the cause of large moments may be noted from the close agreement between the stresses for the same conditions based on the writer's analysis (which assumes uniform pressure distribution below the horizontal diameter) and those obtained by the author. In fact, from the vertical components of reactions in Col. 11, Table 11, the pressure distribution obtained on the horizontal projections of the corresponding segments is greatest at the vertical axis of the tunnel and less toward the sides. Such pressure distribution causes greater moments at J than does uniform upward pressure for the same total load. Therefore, the relatively small stresses, obtained by either the author's method or the writer's method, are due to the horizontal passive pressures involved in both methods.

When a tunnel section is considered or finally decided upon for a project, it may be necessary to obtain the stresses for various conditions of head of water, height of earth cover, and nature of soil, for which different soil constant values may have to be assigned. To further simplify the process of obtaining the stress at point J for various conditions, a chart similar to Fig. 23 would be useful. This chart was prepared for the ring having a sectional area of 27.6 sq in. and, on the basis of the author's loading assumptions, without corrections.

The notation is defined as follows:  $\sigma_A$  = bending stress for active loading, including side earth pressure, cover =  $H_a$  (see Fig. 21);  $\sigma_B$  = bending stress caused by the water load, without considering buoyancy (to conform with the author's assumption), head of water =  $H_b$ ;  $\sigma_D$  = bending stress due to all active loading ( $\sigma_D = \sigma_A - \sigma_B$ );  $\sigma_a$  = compressive stress due to the thrust caused by active earth pressure only;  $\sigma_b$  = compressive stress in addition to  $\sigma_a$ , due to the thrust caused by water, head of water =  $H_b$ ; and  $\sigma_c$  = compressive stress in addition to  $\sigma_a$ , due to the thrust caused by water, head of water =  $H_c$ .

The  $\sigma_R$ -lines, based on Eq. 36, are used to obtain the final bending stresses after taking account of the bending stresses due to the passive pressures corresponding to the indicated soil constants. The other lines in Fig. 23 were obtained by selecting the proper moment and thrust coefficients from Table 18. From Fig. 23 the final stress can be obtained readily after assuming a soil constant, or the soil constant necessary to limit the stress to a certain amount may be determined.

*Problem 1.*—Under the conditions stipulated by Mr. Bull in Section 21 (see conditions B, Table 19), let  $H_a$  (Fig. 21) = 36 ft and  $H_b$  = 27 ft. Entering Fig. 23 at  $H$  = 36 ft, the vertical dropped to the  $\sigma_a$ -line in Fig. 23(a) intersects at  $\sigma_a$  = - 0.4 kip per sq in.; and, extended vertically to the  $\sigma_A$ -line in Fig. 23(b), intersects at  $\sigma_A$  = 23.7 kips per sq in. Again entering Fig. 23 at  $H$  = 27 ft, the vertical dropped to the  $\sigma_b$ -line in Fig. 23(a) intersects at  $\sigma_b$  = - 0.6 kip per sq in.; and, extended vertically to the  $\sigma_B$ -line in Fig. 23(b), intersects at  $\sigma_B$  = - 13.2 kips per sq in. The bending stress  $\sigma_D$  due to all

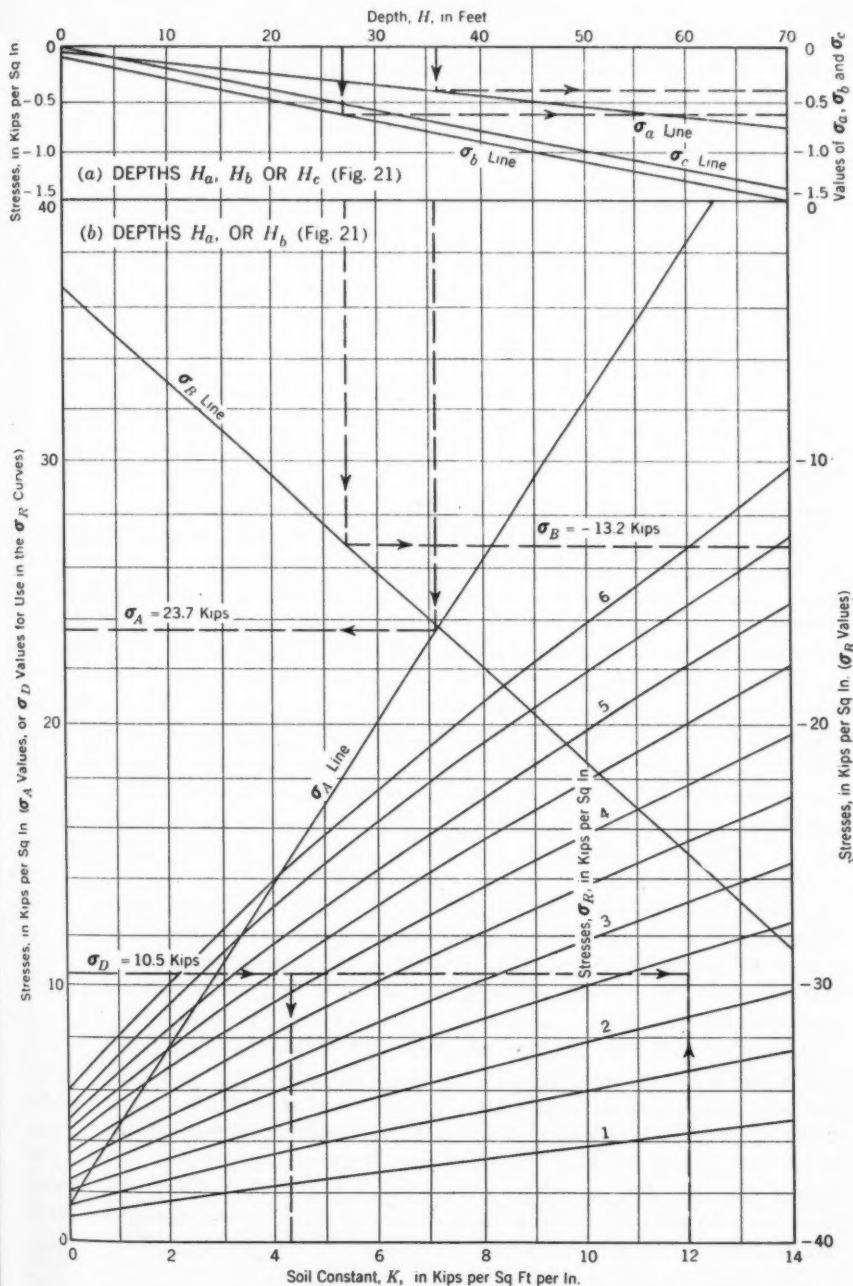


FIG. 23.—COMPREHENSIVE CHART FOR DETERMINING THE STRESS AT JOINT J FOR A TUNNEL RING HAVING AN AREA OF 27.6 SQ IN.

active loading is  $23.7 - 13.2 = 10.5$  kips per sq in. In Fig. 23(b), the intersection of a horizontal from  $\sigma_D = 10.5$  and a vertical from  $K = 12$  gives  $\sigma_R = 2.3$  kips per sq in. The total of  $\sigma_a + \sigma_b = -0.4 + -0.6 = -1.0$  kip per sq in., and  $\sigma_J$ , without considering air pressure, is  $2.3 - 1.0 = 1.3$  kips per sq in. Adding the air-pressure effect,  $+0.69$  kip per sq in., the final value of  $\sigma_J = +1.3 + 0.69 = 1.99$  kips per sq in., which checks the stress given in Table 19. For a total stress  $\sigma_J$  limited to 4.0 kips per sq in.,  $\sigma_R$  could be as great as  $4.0 + 1.0 - 0.69 = 4.31$  kips per sq in. Entering Fig. 23(b) at  $\sigma_D = 10.5$  kips per sq in. and drawing a line horizontally to  $\sigma_R = 4.31$  yields (vertically)  $K = 4.4$ , which demonstrates how the necessary soil constant may be obtained so that the stress shall not exceed a certain value.

SIGVALD JOHANNESSON,<sup>21</sup> M. AM. SOC. C. E.<sup>21a</sup>—The use of cast iron for tunnel linings was conceived by Marc Isambard Brunel in 1818, but it was not until the Tower Subway under the Thames River in London, England, was built in 1869 that a tunnel was constructed with cast iron as lining material. Since then cast-iron tunnel lining has been used extensively for shield-driven tunnels, particularly in Great Britain and the United States.

Until comparatively recently cast-iron linings have been proportioned almost entirely by judgment, checked perhaps by static stress analysis. Since about 1920, however, various attempts have been made to develop a rational method for determining the stresses in tunnel linings, and the author is to be congratulated for having made a most important step in this development.

Based on his formulas the author finds that an excessive volume of metal is used in the present cast-iron linings. Although this may be true theoretically, it is still questionable whether cast-iron linings could be materially decreased in weight. The author gives two reasons why this may not be done: The first is the difficulty of casting large segments of slender shape; and the second is the need for a large cross section to resist the pressure of the shoving jacks. Also, cast-iron linings are cast in segments bolted together at the horizontal flanges, the deflections of which, under circumferential tension, confound the theoretical conclusion; and forces originating in the interior of the tunnels, such as an explosion or a collision, might cause the tunnel to collapse if the cast-iron lining were as thin as the theoretical stress computations would indicate to be safe. These conditions have been, and must be, considered in proportioning the lining as long as the present type of cast-iron lining is used.

However, with the development of the art of making, shaping, and fabricating steel, the present type of cast-iron lining should, and will, become obsolete, and it will be superseded by linings of steel (as suggested by the writer in 1922)<sup>22</sup>, provided that the natural hesitancy of tunnel designers to break with precedent may be overcome. One of the greatest obstacles to change has been the uncertain knowledge of the stresses to which a tunnel lining may be subjected. The author's contribution to the determination of these stresses will greatly facilitate the solution of this problem.

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<sup>21a</sup> Received by the Secretary January 23, 1945.

<sup>22</sup> "Shield and Compressed Air Tunneling," by B. H. M. Hewett and S. Johannesson, 1st Ed., McGraw-Hill Book Co., Inc., New York, N. Y., 1922.

# AMERICAN SOCIETY OF CIVIL ENGINEERS

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## DISCUSSIONS

### TRANSPORTATION OF SUSPENDED SEDIMENT BY WATER

#### Discussion

BY WESTON GAVETT, AND BERARD J. WITZIG

WESTON GAVETT,<sup>47</sup> Assoc. M. Am. Soc. C. E.<sup>47a</sup>—Velocity distribution profiles, obtained from careful observations of velocities in a wide open channel with known bottom roughness, are an important part of this comprehensive paper.

The author states (see heading, "Results: Velocity Distribution") that "In all runs the velocity distribution fits the logarithmic law very well, although near the bottom there is a systematic deviation from the law, which is not understood." Figs. 6 and 7 are plotted with semilogarithmic coordinates in accordance with the von Kármán velocity defect law (Eq. 14), which the author states (see heading, "Previous Work") " \* \* \* has been shown by J. Nikuradse<sup>9,10</sup> to apply to pipes and by G. H. Keulegan<sup>11</sup> to apply also to open channels."

The general acceptance of the von Kármán law which the author endorses in this and in an earlier paper<sup>29</sup> may discourage interest in the exponential formula of the form,

$$u = u_{\max} \left( \frac{y}{y_{\max}} \right)^n \dots \dots \dots (29)$$

which, although not universal, seems to give better agreement with observations.

NOTE.—This paper by Vito A. Vanoni, Assoc. M. Am. Soc. C. E., was published in June, 1944, *Proceedings*. Discussion on this paper has appeared in *Proceedings*, as follows: December, 1944, by Ralph W. Powell, and E. R. Van Driest.

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<sup>47a</sup> Received by the Secretary January 5, 1945.

<sup>9</sup> "Gesetzmässigkeiten der turbulenten Strömung in Glatten Rohren," by J. Nikuradse, *Forschungsheft 559*, Vereines Deutscher Ingenieure, Vol. 3, 1932, p. 36.

<sup>10</sup> "Strömungsgesetze in Rauhen Rohren," by J. Nikuradse, *Forschungsheft 561*, Vereines Deutscher Ingenieure, Vol. 4, 1933, pp. 1-22.

<sup>11</sup> "Laws of Turbulent Flow in Open Channels," by Garbis H. Keulegan, *Journal of Research*, National Bureau of Standards, U. S. Dept. of Commerce, Vol. 21, 1938, pp. 707-741.

<sup>29</sup> "Velocity Distribution in Open Channels," by Vito A. Vanoni, *Civil Engineering*, June, 1941, pp. 356-357.

Nikuradse<sup>45</sup> showed that his velocity distribution curves for smooth pipe followed the exponential law and that  $n$  varied with the Reynolds number. Prandtl<sup>48</sup> has mentioned the seventh-power law based on Blasius' experimental law of pressure drop and has shown that, although the seventh-power law breaks down for Reynolds' numbers above about 50,000, similar derivations may be made for other powers. The writer has plotted a number of velocity profiles from various sources on log-log paper and has found that the greater number show good agreement with the exponential formula.

The velocity defect law has the advantages of being universal and of locating the approximate velocity profile when the friction velocity is known, without regard to roughness.

The exponential formula also has merits: (1) In giving a more accurate expression for velocity distribution and (2) in providing a simple expression for the velocity distribution in a pipe or channel and perhaps also for that in a plate or side of a ship. To apply the exponential formula, it is necessary to know the appropriate value of  $n$ , which appears to vary in accordance with definite laws that should be determinable from sufficient experimental data. For smooth pipe,  $n$  varies with the Reynolds number and decreases as  $R$  increases. It also varies in smooth pipe with  $\delta$ , the laminar thickness, showing an exponential relationship. If  $n$  is plotted against  $\delta$ , the laminar thickness for smooth pipe, and  $\epsilon$ , the equivalent sand size for rough pipe, on log-log paper, the  $(n-\epsilon)$ -curves for rough pipe appear to intercept the  $(n-\delta)$ -curve for smooth pipe at points determined by a value of  $\tau/\epsilon$  and to branch off with slopes greater than the smooth pipe curve.

The author states (see heading, "Results: Velocity Distribution") that "The fact that the velocity profiles fit the logarithmic law so well means that the average velocity will always occur at the relative depth  $\frac{y}{y_m} = 0.368$ ."<sup>49</sup> If the exponential law is valid, there will be some variation of  $y/y_{\max}$  with  $n$  because

$$\frac{u_{\text{avg}}}{u_{\max}} = \frac{1}{n+1} \dots \dots \dots (30a)$$

and

$$\frac{y_{\text{avg}}}{y_{\max}} = \left( \frac{1}{n+1} \right)^{1/n} \dots \dots \dots (30b)$$

By replotting the points from Figs. 6 and 7 on log-log paper, the writer obtained a mean value of  $n$  of 0.145 for runs 1A, 2B, and 3C with a bottom roughness of 0.47-mm sand and no suspended load. For bottom sand size of 0.88 mm and no suspended load (runs 14A, 14B, and 21C), the value of  $n$  was 0.175. For these values of  $n$ , the mean velocity depths from Eq. 30b are as follows:

Description	Runs 1A, 2B, and 3C	Runs 14A, 14B, and 21C
Bottom sand size, in millimeters . . . . .	0.47	0.88
Roughness coefficient $n$ . . . . .	0.145	0.175
Relative depth ratio $\frac{y_{\text{avg}}}{y_{\max}}$ . . . . .	0.393	0.398

<sup>45</sup> "Gesetzmässigkeiten der turbulenten Strömung in Glatten Röhren," by J. Nikuradse, *Forschungsheft 356*, Vereines Deutscher Ingenieure, Vol. 3, 1932, pp. 1-36.

<sup>48</sup> "Applied Hydro- and Aeromechanics," based on lectures of L. Prandtl, by O. G. Tietjens, McGraw-Hill Book Co., Inc., New York, N. Y., 1934, p. 70.

The variation in  $y_{avg}/y_{max}$  is not great, but, for the extreme values of  $n$  for smooth and rough surfaces of about 0.10 and 0.36, the corresponding values of  $y_{avg}/y_{max}$  would be 0.385 and 0.425.

For pipe, the author<sup>29</sup> has given a constant mean distance of  $0.223 r_0$ . The following expressions are derived from the exponential formula for pipe:

$$\frac{u_{avg}}{u_{max}} = \frac{2}{(n+1)(n+2)} \dots \dots \dots (31a)$$

and

$$\frac{y_{avg}}{r_0} = \left[ \frac{2}{(n+1)(n+2)} \right]^{1/n} \dots \dots \dots (31b)$$

For values of  $n$  of 0.1 and 0.36, the corresponding values of  $y_{avg}$  would be  $0.237 r_0$  and  $0.269 r_0$ .

The author's data indicate that the bottom sand size, equivalent to that used by Nikuradse, is smaller than the actual size of sand used, possibly because a part of the sand was embedded in the coating.

The average velocities computed by Eq. 30a check closely with the average velocities given by the author as computed from the velocity defect law, indicating that the von Kármán law may be correct for the average velocity. If so, the formula of B. A. Bakhmeteff,<sup>49</sup> M. Am. Soc. C. E., is simple and convenient:

$$\frac{u_{max} - u_{avg}}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{3}{2k} \dots \dots \dots (32)$$

in which  $\sqrt{\frac{\tau_0}{\rho}}$  is the critical value of  $u$ . Substituting  $k = 0.4$  in Eq. 32,

$$\frac{u_{max} - u_{avg}}{\sqrt{\frac{\tau_0}{\rho}}} = \frac{3}{2} \times \frac{1}{0.4} = 3.75. \quad \text{The same value of 3.75 results from substituting } 0.223 \text{ for } y/r_0 \text{ and } k = 0.4 \text{ in Eq. 14.}$$

Prandtl's value for this constant according to Professor Bakhmeteff is 4.07, which agrees better with values computed from Nikuradse's data.

BERARD J. WITZIG,<sup>50</sup> ASSOC. M. AM. SOC. C. E.<sup>50a</sup>—An excellent service to the science of hydraulics has been rendered by Mr. Vanoni in enlarging and completing his previous paper on sediment transportation,<sup>30</sup> and he is to be commended for a careful exposition and presentation of details leading to logical conclusions. The problem of sediment suspension and transportation by water involves fundamental principles that are only imperfectly understood, and only by painstaking research will limited existing knowledge of the phenomena be expanded.

<sup>49</sup> "The Mechanics of Turbulent Flow," by Boris A. Bakhmeteff, Princeton Univ. Press, Princeton, N. J., 2d Ed., 1941, Eq. 116, p. 70.

<sup>50</sup> Engr. (Civ.), U. S. Engr. Office, Buffalo, N. Y.

<sup>50a</sup> Received by the Secretariat January 15, 1945.

<sup>30</sup> "Some Experiments on the Transportation of Suspended Load," by Vito A. Vanoni, *Transactions, Am. Geophysical Union*, Pt. III, 1941, pp. 608-621.

In the "Introduction," the author mentions two essentially different modes of sediment transportation—"bed load" and "suspended load." As long as it is remembered that the distinction between these modes is merely arbitrary<sup>51</sup> because of the insurmountable difficulties of treating the sediment load of a stream as a whole, there should be no slighting of the possibility that a much larger quantity of sediment in some streams may move as bed load than as suspended load.

The relations and the results given in this paper should not be applied indiscriminately to the total suspended load. The exponential vertical distribution of sediment applies only to the coarse suspended load. However, in some streams, a large part of the total suspended load consists of very fine silt, termed the "wash load,"<sup>52</sup> and is very nearly uniformly distributed over the stream cross section. Any conclusions based on a logarithmic or exponential distribution of the wash load would be misleading or erroneous. Apparently an appreciable wash load did not occur in the author's experiments, in which a carefully graded sand was used. It would be of interest, however, to know Mr. Vanoni's views on the probable influence of a uniformly distributed wash load, in combination with a logarithmically distributed coarse suspended load, on the sediment transfer coefficient.

Although probably of merely academic interest, it seems appropriate to point out that a transitional mode of sediment transportation exists, known as "saltation." In saltation, grains of bed material are lifted momentarily into the moving fluid, then bouncing back on to the bed, possibly dislodging other particles. When the turbulence of the stream exceeds a certain level, these particles move into the suspended load zone. Quantitatively, the saltation load is probably insignificant. Prof. A. A. Kalinske,<sup>53</sup> Assoc. M. Am. Soc. C. E., concluded that, when the water velocity is great enough to cause saltation, the turbulence level is also sufficient to obscure the saltation effect entirely, by placing the material lifted from the stream bed into suspension.

The paper succinctly summarizes modern fluid turbulence theory as applied to sediment suspension. The present state of knowledge and research necessitated the author's initial assumptions that the sediment transfer coefficient,  $\epsilon_s$ , is equal to the momentum transfer coefficient,  $\epsilon_m$ , and that these coefficients are constant over the depth. The experimental results described demonstrate the falsity of the former assumption, but it is not apparent whether they either prove or disprove the latter. The writer considers it probable that some of the apparent discrepancies in the published data, such as the large variation in the values of  $z$  and  $z_1$ , run 17, Table 3, might be due to some unaccountable deviation in the  $(\epsilon - y)$ -relation.

The author explains that the laboratory flume used in his experiments had rubber sides, with apparently a very small roughness coefficient. A narrow flume, however, is bound to exert a pronounced "side-wall effect," especially

<sup>51</sup> "Fluid Mechanics for Hydraulic Engineers," by Hunter Rouse, McGraw-Hill Book Co., Inc., New York, N. Y., 1938, p. 331.

<sup>52</sup> Discussion by Joe W. Johnson of "Sedimentation in Reservoirs," by Berard J. Witzig, *Transactions, Am. Soc. C. E.*, Vol. 109 (1944), p. 1072.

<sup>53</sup> "Criteria for Determining Sand-Transport by Surface Creep and Saltation," by A. A. Kalinske, *Transactions, Am. Geophysical Union*, Pt. II, 1942, pp. 639-643.

where the bottom itself is comparatively smooth, as was the case here and as is shown in Fig. 5. Although Fig. 5 demonstrates that the side-wall effect of the experimental flume extended only about one sixth of the flume width from each side, the writer is of the opinion that such an extensive series of tests should have been made in a wider flume. A fundamental assumption for the two-dimensional analysis of the problem is that the theoretical equations for the vertical distribution of velocity and of sediment are valid only in "wide" channels. In this case, the ratio of depth to width was about 1:5.5, and its recognized effects on the experimental observations are indicated by the fact that all measurements in Table 3 were confined either to the center line of the flume, or to a vertical not more than 0.5 ft from the center line. Laboratory limitations, of course, specify the practicable width of flume, but some increase in the ratio of width to depth should be desirable in future experiments.

The effect of the depth-width ratio in a turbulent and rapidly flowing natural stream is illustrated by discharge measurements made in 1943 under the writer's direction near the head of the Cascades, in the American channel of the Niagara River at Niagara Falls, N. Y. At the time of the measurements, the average depth was about 6 ft; the width at the gaging section, about 480 ft; the measured discharge, 16,150 cu ft per sec; and the mean velocity, 7.5 ft per sec. Vertical velocity traverses made with a current meter showed that a generally exponential gradient existed, with the maximum velocities occurring at the surface. These measurements also indicated that the mean velocity in a vertical occurred, on the average, at the 0.55 depth from the surface, instead of at the 0.6 depth commonly assumed in discharge measurements. The logarithmic law places the mean at the 0.632 depth from the surface. It seems probable that such variation in field observations from theoretical assumptions and laboratory results must place in doubt the validity of generally applying the latter to the large-scale phenomena existing in natural streams.

The writer was particularly interested in the analysis of the effect of sediment loads on the roughness coefficient of the flume. Runs 18, 20, and 21, Table 4, however, apparently indicate some inconsistencies. For the clear water run in each case, Manning's  $n$  was 0.0118, 0.0121, and 0.0128, respectively; whereas, with a sediment concentration of 1.2 g per liter,  $n$  was 0.0107, 0.0116, and 0.0116, respectively. Runs 18 and 21 showed a ratio of 1.10 for clear flows compared with silt flows, but run 20, under practically identical conditions, gave a ratio of 1.04. Can this apparent inconsistency be explained by the change in slope and velocity, or should it be ascribed to the difficulties of laboratory measurement?

It would be of interest to find some explanation of the mechanism whereby the channel roughness is decreased by the presence of sediment in the fluid. May not the lamina of high concentration fill the rough projections of the channel bed, thereby acting as a lubricating film and decreasing the apparent roughness coefficient? Merely as a speculation, it is also suggested that an increase in silt load may decrease the apparent roughness at a diminishing rate to a certain point, and that thereafter a further increase in silt content may produce a result similar to an increase in viscosity, with greater resistance to flow.

The author suggests that one of the reasons for the difference between the momentum and sediment transfer coefficients is the "slip" between the fluid and the sediment particles. The writer considers that this "slip" is due in part to the inertia and the "drag" of the particles. The difference in the effect of fine and coarse sediments on  $\epsilon_s$  compared with  $\epsilon_m$  might reasonably be expected. Small particles, with a relatively larger surface area per unit of mass as compared with larger particles, should naturally be more readily moved or kept in suspension by the random turbulence; the "slip" effect also should be less because of the smaller mass and drag; and molecular impact should be more pronounced.

It is suggested that discrepancies between  $z$  and  $z_1$ , as well as the consistent variation actually demonstrated, might be caused by secondary, or more properly "tertiary," circulation in the third (horizontal) dimension. If such variations appear in carefully controlled laboratory tests, even wider variations may be expected in natural streams, where complex circulations are induced by winds at the surface, bends in alignment, tributary and surface inflow, and irregularities of the sides and bed.

The description of the horizontal distribution of sediment in segregated bands and clouds omits reference to any observed vertical segregation. Did such vertical separation exist, and, if so, did the sediment tend to follow the fluid filaments of highest velocity? The writer's observations of silt-laden natural streams indicate that sediment flows appear to move in "slugs" or clouds, but he has not noticed band-like movements. It is probable that these may not occur in natural streams because of the large degree of turbulence. With reference to Fig. 15(c), it is not readily apparent how the secondary circulation is derived from the velocity and sediment diagrams.

It is questionable whether the increase in velocity accompanying silt flows results in an actual decrease in the entrainment forces exerted on bed particles. An increase in stream velocity causes increased erosion when the stream is not burdened to capacity with its sediment load. When the stream is fully charged with sediment, it seems probable that the concentration near the bottom may be so great that no net increase in load may result, although a continual interchange takes place. Such a stable condition may be due, not so much to the theoretical decrease in shear accompanying the smaller depth and greater velocity, as to the fact that the full transporting capacity is already in use.

The writer considers that the author has ably and amply demonstrated previously anticipated effects of sediment loads on the transfer coefficient, and that his experiments indicate the relative magnitude of these effects for the range of concentrations and sizes of sediments studied. The apparent effect on channel roughness or resistance suggests that further intensive research on this particular aspect of the problem would be of interest economically, as it may appreciably affect practical design of pipe lines or flumes for commercial or industrial transportation of materials in suspension.

The author has shown that a very complex problem in hydraulics is susceptible, at least in part, to rational and experimental analysis. Other experimenters should find encouragement in his report of progress.

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Founded November 5, 1852

## DISCUSSIONS

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### EVAPORATION FROM A FREE WATER SURFACE

#### Discussion

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BY CARL ROHWER

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CARL ROHWER,<sup>4</sup> M. AM. SOC. C. E.<sup>4a</sup>—Observations on the evaporation from free water surfaces in pans are usually made either to discover the fundamental laws showing the relation between the meteorological data and evaporation or to make available information from which it will be possible to estimate the loss by evaporation from large bodies of water such as lakes, reservoirs, and streams.

Because of the complex nature of evaporation phenomena and the many dependent variables involved, investigators have resorted to various expedients when developing evaporation formulas to simplify the analysis of the effect of the influencing factors. Since the laws governing certain similar physical phenomena such as turbulence, heat conduction, and mass transfer have been fairly well established, the analogies between these phenomena and evaporation are sometimes used as a basis for attacking the problem of determining the laws of evaporation. This is the method adopted by the author. However, evaporation laws cannot be developed by analogy alone because there are certain constants involved which must be determined experimentally, and for which carefully controlled evaporation observations are necessary.

The evaporation formula (Eq. 5) based on analogy between heat transfer and diffusion developed by the author is represented by the dashed line in Fig. 5. Evaporation data obtained by the author, by means of the optical interferometer designed by him, when plotted on the same figure, result in the two full lines that meet at the point where the vapor-concentration difference is  $4.2 \times 10^{-4}$  lb per cu ft. Since the line representing Eq. 5 and the lower part of the line representing the experimental data both pass through the origin, Eq. 5 could be made to conform to the experimental data by multiplying by a constant. However, Eq. 5 cannot be made to conform to the remainder of the experimental data by a simple transformation. The fact that there is a break in the line indicates that a different law applies beyond this point.

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NOTE.—This paper by G. H. Hickox was published in October, 1944, *Proceedings*.

<sup>4</sup>Irrig. Engr., Div. of Irrig., SCS, U. S. Dept. of Agriculture, Colorado Experiment Station, Fort Collins, Colo.

<sup>4a</sup>Received by the Secretary January 17, 1945.

As stated by the author, Eq. 5 applies only to still air conditions which probably are never obtained when conducting evaporation experiments even under controlled laboratory conditions. Although all external disturbances are excluded, any difference in temperature or density will cause the air to move. This is confirmed by the experience of the writer in conducting evaporation experiments under controlled conditions in the laboratory at Fort Collins (16a),<sup>4b</sup> although the exact significance of the phenomenon was not recognized at the time. The evaporation was at a maximum when the water was warmer than the air. When this occurred, the air in contact with the water would be warmer than the surrounding air and would rise, carrying with it a certain amount of water vapor which would have had to be removed by diffusion if the air were absolutely still. When the water was colder than the air, the density gradient was reversed and the air would remain in contact with the water surface because it was held there by the rim of the tank. When this occurred the water vapor could escape only by diffusion, which is a slow process. These explanations, however, do not account for the fact that the observed evaporation below the break in the curve (Fig. 5) is about four times that computed by Eq. 5. In fact, the discrepancy beyond the break in the curve is so great that it seems doubtful whether the analogy of mass transfer is valid.

According to the author's analysis the evaporation should vary as the 0.75 power of the wind velocity. The results obtained by several investigators are shown plotted in Fig. 8. Although the total effect of wind found by these investigators varies considerably, with one exception the results conform to the 0.75 power at the higher velocities. Whether this is a fundamental difference or whether it is due to the uncertainty of wind measurements at low velocities is not known. The author states that F. Graham Millar was able to approximate actual evaporation records on the basis of turbulence and eddy diffusivity, but could correlate the writer's data only on the assumption that the anemometer was in error. Cup anemometers probably are not the most accurate devices for measuring wind velocity, but they are almost universally used and for this reason they were adopted for the experiments at Fort Collins. Although the indicated velocities differ from the true velocities, particularly at high velocities, similar differences would occur whenever the anemometers were used and consequently comparable results would be obtained.

The effect of pan diameter on evaporation has been plotted by the author in Fig. 9, showing that beyond 10 ft the diameter has very little effect on the evaporation. This is the generally accepted conclusion. However, the author's formula for evaporation into air in motion, Eq. 15b, indicates that the evaporation is inversely proportional to the fourth root of the diameter of the pan. According to this formula, the evaporation from a 10-ft pan would be more than five times as great as that from a pan 9,000 ft in diameter under similar conditions instead of about 10% greater as indicated in Fig. 9.

Fig. 10 shows the relation between the barometric pressure and the relative evaporation at different altitudes based on data obtained by the writer, recomputed in accordance with Eq. 20. Fig. 13 shows the same data and in addition the relative evaporation as computed by the writer (16b). It will be

<sup>4b</sup> Numerals in parentheses, thus: (16a), refer to corresponding items in the Bibliography (see Appendix I of the paper), and at the end of discussion in this issue.

observed that the points fall much closer to a line with a slope of  $-0.25$  than do those computed by the author. This indicates that the empirical analysis used by the writer gives results that are more nearly in accord with the facts

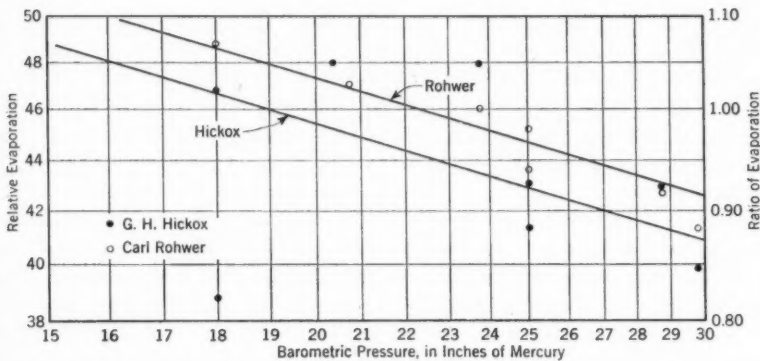


FIG. 13.—RELATION OF EVAPORATION TO BAROMETRIC PRESSURE BASED ON DIFFERENT ANALYSES OF THE SAME DATA

than the analysis based on the analogy between evaporation and other physical phenomena.

The Sub-Committee on Evaporation of the Special Committee on Irrigation Hydraulics of the Society (2) adopted 0.70 as the ratio of the annual evaporation from a Weather Bureau pan to that from a reservoir. The author states (see heading, "Introduction") that the ratio is probably not constant through-

TABLE 5.—MONTHLY RATIOS FOR THE CONVERSION OF THE EVAPORATION FROM A WEATHER BUREAU PAN TO THE EVAPORATION FROM OTHER WATER BODIES

Month	(a) LAKE ELSINORE, CALIFORNIA				(b) 12-FT SUNKEN PAN AT FULLERTON, CALIF.				
	1939	1940	1941	Mean	1936	1937	1938	1939	Mean
January	0.90	0.85	0.65	0.80	....	0.65	0.65	0.66	0.65
February	0.69	0.66	0.51	0.62	....	....	0.67	0.71	0.69
March	0.79	0.58	0.68	0.68	0.77	0.73	0.75	0.80	0.76
April	0.59	0.68	0.74	0.67	0.79	0.79	0.79	0.81	0.80
May	0.72	0.66	0.66	0.68	0.80	0.82	0.78	0.83	0.81
June	0.78	0.77	0.75	0.77	0.81	0.79	0.85	0.82	0.82
July	0.75	0.72	0.75	0.74	0.79	0.81	0.81	0.83	0.81
August	0.78	0.69	0.87	0.78	0.79	0.83	0.79	0.81	0.80
September	0.84	0.84	0.92	0.87	0.79	0.78	0.78	0.71	0.76
October	0.94	0.90	0.97	0.94	0.75	0.74	0.78	0.74	0.75
November	0.90	1.08	0.91	0.96	0.68	0.75	0.72	0.76	0.73
December	0.93	1.11	0.78	0.94	0.57	0.72	0.65	0.74	0.67
Mean (weighted)	(0.78)	(0.76)	(0.78)	(0.77)	(0.77)	(0.78)	(0.77)	(0.78)	(0.78)

out the year. Although this view is proved by the writer's experiments at Fort Collins (16c), it is doubtful whether it is worth while to use a different ratio for each month because conditions vary from year to year on account of seasonal differences. Table 5 (computed by the writer from data collected by A. A. Young, Assoc. M. Am. Soc. C. E. (43)(44), for the Division of Irriga-

tion, Soil Conservation Service, U. S. Department of Agriculture) shows that the ratio reached a maximum during the cold months in the fall and winter at Elsinore, Calif., whereas at Fullerton, Calif., the maximum ratio occurred during the warm months of the year. The mean values for these two ratios for the entire year are, respectively, 0.77 and 0.78 instead of 0.70 as recommended by the writer and by the Sub-Committee on Evaporation of the Special Committee on Irrigation Hydraulics of the Society. There was little change in the ratios from year to year. These results support the author's contention that the ratios would not be the same under Southern California conditions as they would be in areas where the evaporation is very small during the winter months.

The optical interferometer developed by the author is an ingenious device for measuring very small rates of evaporation with precision. It is unfortunate that the experiments with this device were limited to a narrow range of conditions. A series of observations under controlled conditions in the laboratory on the evaporation from pans of different sizes into air moving at definite velocities would have made it possible for Mr. Hickox to check the accuracy of Eq. 15b without making assumptions as to the conditions under which the tests were made.

A study of the device raises the questions of whether the plate glass reflector F, in Fig. 3, would not reflect the beam of light from both surfaces of the plate glass and of whether, as a result, two sets of interference bands would appear to the observer. The one from the interior face of the glass plate would be less distinct, but, when the dark bands of one set were superimposed over the light band of the other, the difficulty of counting the bands would be increased. This possibility is mentioned because of a similar difficulty experienced by the writer in making evaporation observations by the use of the optical lever. Until the plate glass mirror first used was replaced by a metal mirror with only one reflecting surface, two distinct images of the scale on which the evaporation was read appeared in the telescope. The use of a metal mirror completely eliminated the second image.

After studying the paper the reader cannot help but feel that there is considerable doubt as to whether the analogy used is suitable for interpreting evaporation phenomena. At least the available data do not fit the theory very well. The writer does not wish to leave the impression that he is attempting to discredit Mr. Hickox's work even though the results are not conclusive. Much of the work on evaporation has been on an empirical basis and any attempt at a rational analysis of the problem should be commended.

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- (16) "Evaporation from Free Water Surfaces," by Carl Rohwer, *Bulletin No. 271*, U. S. D. A., December, 1931. (a) p. 9. (b) pp. 34-42. (c) p. 71.
- (43) "Evaporation Studies at Lake Elsinore, California," by A. A. Young, Div. of Irrig., SCS, U. S. D. A., 1942 (mimeographed).
- (44) "Investigation of Evaporation from a Screened Pan and Development of a Coefficient for the Conversion of Pan Evaporation to Lake Evaporation," by A. A. Young, Div. of Irrig., SCS, U. S. D. A., 1942 (mimeographed).

# AMERICAN SOCIETY OF CIVIL ENGINEERS

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## DISCUSSIONS

### ANALYSIS OF THE GENERAL TWO-DIMENSIONAL FRAMEWORK

#### Discussion

BY A. A. EREMIN

A. A. EREMIN,<sup>16</sup> Assoc. M. Am. Soc. C. E.<sup>16a</sup>—An interesting method for computing stresses and deformations in continuous frames is presented by Mr. Nubar. Like Prof. O. Mohr,<sup>17</sup> Mr. Nubar considers the geometric and loading factors as forces applied at the joints of the frame, although, regrettably, his illustrative example (Fig. 13) fails to demonstrate any superior simplicity in practical application. By graphical methods, the moments in the frame, shown in Fig. 13, can be determined with less effort and time.

Graphically, the writer would proceed in two steps:<sup>18</sup> The first step is to determine the physical properties of the frame by locating the characteristic points, termed "central points," "inflection points," and "elastic points." The relative location of the characteristic points is determined by constructing a "three-line polygon," as described briefly by the writer elsewhere.<sup>19</sup> The second step is to determine the moments produced by the given loading and the distortion of the frame.

Referring to Fig. 20, which is an analysis of the frame shown in Fig. 13: The central moments in member CD with a load of 40 kips are  $M_1 = M_2 = 80$  kip-ft. If the joints are fixed horizontally, the moments in the frame are determined by drawing "cross lines," as in Fig. 20(c). A distortion diagram with a unit displacement at joint C is shown in Fig. 20(b). The basic bending moments due to distortion in member CD are

$$M_{CD} = N a k_1 = 42.3 k_1 \dots \dots \dots (66a)$$

NOTE.—This paper by Yves Nubar, M. Am. Soc. C. E., was published in April, 1944, *Proceedings*. Discussion on this paper appeared in *Proceedings*, as follows: June, 1944, by Leon Beskin; and September, 1944, by Jaroslav Polivka.

<sup>16</sup> Associate Bridge Engr., Bridge Dept., Div. of Highways, State Dept. of Public Works, Sacramento, Calif.

<sup>16a</sup> Received by the Secretary January 2, 1945.

<sup>17</sup> "Secondary Stresses in Bridges," by Cecil Vivian von Abo, *Transactions*, Am. Soc. C. E., Vol. 89 (1926), p. 31.

<sup>18</sup> "Analysis of Continuous Frames by Graphical Distribution of Moments," by A. A. Eremín, Sacramento, Calif., 1943 Ed. Published by the author.

<sup>19</sup> *Proceedings*, Am. Soc. C. E., November, 1944, p. 1490.

and

$$M_{DC} = N b k_1 = 78.2 k_1 \dots \dots \dots (66b)$$

in which

$$N = \frac{600 K \Delta}{L (L - a - b)} \dots \dots \dots (67)$$

and  $k_1$  is a distortion coefficient;  $a$  and  $b$  are the distances to the points of inflection (shown in the parentheses, Fig. 20(a));  $\Delta$  is transverse displacement of member ends (Fig. 20(b)); and  $K (= I/L)$  is the basic stiffness of member,

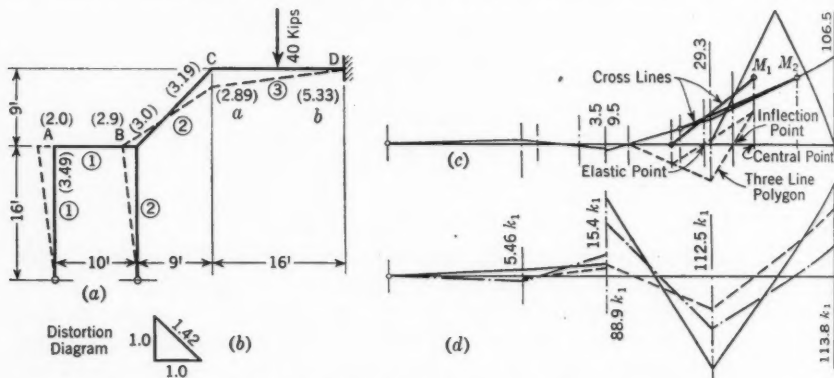


FIG. 20.—GRAPHIC DISTRIBUTION OF MOMENTS

shown in the circles in Fig. 20(a). The numerical factor 600 in Eq. 67 is introduced for convenience of computations. The basic moments in each member are determined by Eqs. 66 and 67 and distributed, as shown in Fig. 20(d).

As a condition of static equilibrium, the reaction at joints E, F, and D is:

$$H_E + H_F - H_D = 0 \dots \dots \dots (68)$$

Using the moments in Figs. 20(c) and 20(d) to determine the values of  $H_E$ ,  $H_F$ , and  $H_D$  in Eq. 68, a value of  $k_1 = 0.48$  is computed.

Superposing the moments in Fig. 20(c) and the moments in Fig. 20(d), the final bending moments in the frame are  $M_{DC} = 161.5$  and  $M_{BF} = 29.1$ , which differ slightly from those computed by the algebraic formulas introduced by the author.

Contrary to Mr. Nubar's method, the operations in the graphic distribution of moments may easily be visualized. Therefore, the moments may readily be checked. Furthermore, graphics may be used in checking the analytical computations of moments. The constant  $C$  used by the author in Eq. 35a may be checked by the inflection point distance in the graphical method.

Thus, for member CD, Fig. 20(a):  $C_{3D} = \frac{a}{L - a} = 0.22$ , which is the same as that computed by the author. Computation of the inflection point distances,  $a$  and  $b$ , in members with variable sections may be simplified by using the constants given by the writer elsewhere.<sup>18</sup>